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Introduction

Because that for the circular relations turned out well, I have decided to complete all of the elementary functions and identities of the equivalent hyperbolic forms. In this instance I think that it is important that, if for nothing else, that these functions are fully tabulated for future reference for both interest and potentially tabulation in reference handbooks.

Centripetal force is the inward normal force of circular motion, of both particle and rigid body dynamics. The double crosses, and hence the double cross hyperbolics, are the centripetal forces [the $(\hat{k} \times)^2$ term]. [as personal opinion:] It is also the force that pulls planes and satellites back to earth (the gravitational force). For this reason, I can confidently say that the double cross hyperbolics are definitely not trivial despite being elementary functions (the ancillary functions for calculus in R^3). The inward direction of the normal force of circular motion is a physical explanation as to why there are negatives on the **sinkk**, **sinjj**, and **sinii** terms when coupled with $\hat{k} \times \hat{k} \times$, $\hat{j} \times \hat{j} \times$, and $\hat{i} \times \hat{i} \times$ in **expkk**, **expjj**, and **expii**. They are also reflections.

In the document *Circular Relations of $i = \hat{k} \times$* I determined and tabulated all of the elementary functions of the basis operator sines, cosines, and tangents: **sink** (“sine-kay”), **sinj** (“sine-jay”), **sini** (“sine-eye”) for sine, **cosk** (“cos-kay”), **cosj** (“cos-jay”), **cosi** (“cos-eye”) for cosine, and **tank** (“tan-kay”), **tanj** (“tan-jay”), and **tani** (“tan-eye”) for tangent. I did so for the **sink** law, and before doing so, it seemed that they were too trivial to state given that they are non-calculus. How it turned out, though, made it clear the tabulation wasn’t trivial. So I am here completing the tabulation of the basis operators for the hyperbolic functions, of which there are two types (in light of this further second hyperbolic type in the *Almighty Cross Product*). The first type is the set of relations pertaining to the double crosses $\hat{k} \times \hat{k} \times$, $\hat{j} \times \hat{j} \times$, and $\hat{i} \times \hat{i} \times$. These are **sinkk** (“sine-kay-kay”), **sinjj** (“sine-jay-jay”), **sinii** (“sin-eye-eye”), **coskk** (“cos-kay-kay”), **cosjj** (“cos-jay-jay”), **cosii** (“cos-eye-eye”), **tankk** (“tan-kay-kay”), **tanjj** (“tan-jay-jay”), and **tanii** (“tan-eye-eye”) which are sign (rather than axis) osculating basis functions and are consequently attenuation factors for the axes of the orthogonal complement (the plane orthogonal to the rotation axis). The second type is the set of hyperbolic functions that are basis operators of the hyperbolic complex numbers $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$, $\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times$, and $\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times$ rather than the double crosses ($\hat{k} \times \hat{k} \times$, $\hat{j} \times \hat{j} \times$, and $\hat{i} \times \hat{i} \times$). These are the hyperbolic functions **sinhk** (“shine-kay”), **sinhj** (“shine-jay”), **sinhi** (“shine-eye”) for sinh, **coshk** (“cosh-kay”), **coshj** (“cosh-jay”), **coshi** (“cosh-eye”) for cosh, and **tanhk** (“than-kay”), **tanhj** (“than-jay”), and **tanhi** (“than-eye”) for tanh, which pertain to axis (rather than sign) osculating basis functions. When the axis osculating hyperbolic sines and cosines are combined in Euler form, these functions produce the hyperbolic exponentials **expkh**, **expjh**, and **expih** and are the hyperbolic equivalents to the rotation matrices of the basis exponentials **expk**, **expj**, and **expi**.

Identities of **sinkk** and **coskk** : Hyperbolic Relations of $\hat{k} \times \hat{k} \times$

Sign Osculating (attenuating) Hyperbolic Sine and Cosine – Relations of the double crosses $\hat{k} \times \hat{k} \times$, $\hat{j} \times \hat{j} \times$, and $\hat{i} \times \hat{i} \times$.

All of the elementary properties of the scalar hyperbolic functions and identities of sinh and cosh (and tanh) hold good for basis hyperbolic sines and hyperbolic cosines (and hyperbolic tangents) similarly as for the circular trigonometric functions for sine and cosine (and tangent). Similarly as the previous document for the circular relations, this document focuses firstly on those of **sinkk** and **coskk** and then by change of base of those of **sinjj** and **cosjj** and similarly **sinii** and **cosii**. Then secondly those of **sinhk** and **coshk** and then by change of base those of **sinhj** and **coshj** and similarly **sinhi** and **coshi**.

Similarly as for the circular relations, I prove the four most important relations for **sinkk** and **coskk** with **expkk** and then change base with (1) **expj**($-\pi/2$) and **expj**($\pi/2$), and (2) **expi**($\pi/2$) and **expi**($-\pi/2$) respectively, to (1) **sinjj** and **cosjj**, and (2) **sinii** and **cosii** [I have essentially just converted over all of the mathematics over to the double crosses which seems like laziness but is rather for purposes consistency between the circular and hyperbolic analogues... the similarities and differences are then more easily seen]. Similarly as for **cosk** and **sink**, all of the proofs for **coskk** and **sinkk** are with the basis operators in terms of the Euler formulae for those operators

$$\begin{aligned}
\coskk\theta &= \frac{1}{2}(\expkk(\theta) + \expkk(-\theta)) \\
&= \frac{1}{2}(\exp(\theta \hat{k} \times \hat{k} \times) + \exp(-\theta \hat{k} \times \hat{k} \times)) \\
&= \frac{1}{2}(e^{\theta \hat{k} \times \hat{k} \times} + e^{-\theta \hat{k} \times \hat{k} \times})
\end{aligned}
\qquad
\begin{aligned}
\sinkk\theta \hat{k} \times \hat{k} \times &= -\frac{1}{2}(\expkk(\theta) - \expkk(-\theta)) \\
&= -\frac{1}{2}(\exp(\theta \hat{k} \times \hat{k} \times) - \exp(-\theta \hat{k} \times \hat{k} \times)) \\
&= -\frac{1}{2}(e^{\theta \hat{k} \times \hat{k} \times} - e^{-\theta \hat{k} \times \hat{k} \times})
\end{aligned}$$

Similarly as for **cosk** and **sink**, these are then proved again with the diadically expanded matrix operators for **coskk** and **sinkk**. This then proves that it all works in matrix bases and provides an easy and efficient way of doing so with a programmable calculator (such as a TI-89 which is my means of choice at the moment given that it is dedicated to the purpose without the need for a compromised os and chip with way too much power to do things that are not in my interest or purpose) or software package such as MATLAB.

Here again I alternatively, and equivalently, change base to the other principal rotation axes and note again that the relations can be proved for the other rotation axes due to similarity invariance with

$$\begin{aligned}
\cosjj\theta &= \frac{1}{2}(\expjj(\theta) + \expjj(-\theta)) \\
&= \frac{1}{2}(\exp(\theta \hat{j} \times \hat{j} \times) + \exp(-\theta \hat{j} \times \hat{j} \times)) \\
&= \frac{1}{2}(e^{\theta \hat{j} \times \hat{j} \times} + e^{-\theta \hat{j} \times \hat{j} \times})
\end{aligned}
\qquad
\begin{aligned}
\sinjj\theta \hat{j} \times \hat{j} \times &= -\frac{1}{2}(\expjj(\theta) - \expjj(-\theta)) \\
&= -\frac{1}{2}(\exp(\theta \hat{j} \times \hat{j} \times) - \exp(-\theta \hat{j} \times \hat{j} \times)) \\
&= -\frac{1}{2}(e^{\theta \hat{j} \times \hat{j} \times} - e^{-\theta \hat{j} \times \hat{j} \times})
\end{aligned}$$

and

$$\begin{aligned}
\cosii\theta &= \frac{1}{2}(\expii(\theta) + \expii(-\theta)) \\
&= \frac{1}{2}(\exp(\theta \hat{i} \times \hat{i} \times) + \exp(-\theta \hat{i} \times \hat{i} \times)) \\
&= \frac{1}{2}(e^{\theta \hat{i} \times \hat{i} \times} + e^{-\theta \hat{i} \times \hat{i} \times})
\end{aligned}
\qquad
\begin{aligned}
\sinii\theta \hat{i} \times \hat{i} \times &= -\frac{1}{2}(\expii(\theta) - \expii(-\theta)) \\
&= -\frac{1}{2}(\exp(\theta \hat{i} \times \hat{i} \times) - \exp(-\theta \hat{i} \times \hat{i} \times)) \\
&= -\frac{1}{2}(e^{\theta \hat{i} \times \hat{i} \times} - e^{-\theta \hat{i} \times \hat{i} \times})
\end{aligned}$$

The formulae for the hyperbolic basis sines (**sinkk**, **sinjj**, and **sinii**) are appended by the relevant principal basis cross ($k \times k \times$, $j \times j \times$, or $i \times i \times$) to avoid the issue of singular (non-full rank) k -cross squared, j -cross squared, and i -cross squared in the denominator.

The sign osculating double cross hyperbolic identities with **sinkk** and **coskk** (k -cross squared equals negative one basis) have suggested pronunciation [as of this time] “sine-kay-kay” and “cos-kay-kay” whilst those of the axis osculating hyperbolics **sinhk** and **coshk** have suggested pronunciation “shine-kay” and “cosh-kay” (and **sinhj** “shine-jay” and **coshj** “cosh-jay” and **sinhi** “shine-eye” and **coshi** “cosh-eye”). These later functions are considered after the double cross (sign osculating) hyperbolics and have significant differences. This suggested pronunciation for the two forms of hyperbolics is consistent with that of “sine-kay” for **sink**, “cos-kay” for **cosk**, and “tan-kay” for **tank**.

Note that a conversion from **sink** and **cosk** to **sinkk** and **coskk** is to make the argument of **sink** and **cosk** a function of $k \times$, similarly as for 2nd year complex analysis and calculus converting sines and cosines to and from the sign osculating (double cross) hyperbolic basis functions. Important to mention is that this does not present a conversion method to the axis osculating hyperbolics **coshk** and **sinhk** although various other results will likely come to light for the axis osculating hyperbolics in this regard (when **coshk** and **sinhk** are made functions of their hyperbolic basis operator $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$).

This is because a **expkk** has a repeated k -hat double cross eigenvalue and an identity (unrotated) matrix of eigenvectors whilst that of **expkh** has a conjugated set of k -hat double cross eigenvalues and a $\pi/4$ rotation matrix set of eigenvectors which changes base for the conjugated eigenvalues to i -hat and j -hat axis osculated. So **expkk** (quadratic roots of the identity I) is a subset of **expk** (the quartic roots of the identity I) whilst that of **expkh** is different in that it is not a subset or subgroup of **expk** as **expkk** is. Here are **expk**, **expkk**, and **expkh** for comparison

$$\expk\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{expkk}\theta = \begin{pmatrix} \cosh\theta - \sinh\theta & 0 & 0 \\ 0 & \cosh\theta - \sinh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{expkh}\theta = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The sign change on scalar exponentials in the diadically expanded form of **expkk**, and those on the sinh scalars that remain on the main diagonal (rather than osculating between $\hat{i}$ -hat and $\hat{j}$ -hat), is due to the property of the double cross $(\hat{k} \times)^2 = -1$. This is the reflection from the square of $\hat{i} = \hat{k} \times$.

cosk, **coskk**, and **coshk** for comparison (**coskk** and **coshk** have the same scalar components)

$$\mathbf{cosk}\theta = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{coskk}\theta = \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{coshk}\theta = \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

sink, **sinkk**, and **sinhk** for comparison (**sinkk** and **sinhk** are the same in terms of scalar components and it is the coupling with $\hat{k} \times \hat{k} \times$ or $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$ which provide the requisite difference between sign and axis osculating hyperbolics)

$$\mathbf{sink}\theta = \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \theta \end{pmatrix}$$

$$\mathbf{sinkk}\theta = \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \sinh\theta & 0 \\ 0 & 0 & \theta \end{pmatrix}$$

$$\mathbf{sinhk}\theta = \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \sinh\theta & 0 \\ 0 & 0 & \theta \end{pmatrix}$$

and coupled with the basis operator ($\hat{k} \times$, $\hat{k} \times \hat{k} \times$, and $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$ respectively)

$$\mathbf{sink}\theta \hat{k} \times = \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ noting the change from the diagonal to anti-diagonal and one sign change (sign and axis osculate)}$$

$$\mathbf{sinkk}\theta \hat{k} \times \hat{k} \times = \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \sinh\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\sinh\theta & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ noting the change of sign on the diagonal (sign osculate)}$$

$$\mathbf{sinhk}\theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \sinh\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sinh\theta & 0 \\ \sinh\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ noting the change from the diagonal (axis osculate)}$$

Sign vs axis osculation is the best way I have at the moment to make sense of the results, in the context of the various (complex) roots of the identity I (ignoring for arguments sake, and for the time being, the difference between the singular and full rank roots/complex numbers/rotation operators).

Periodic properties of **coskk** and **sinkk** (and **tankk**)

Firstly, angle reversal for the basis circular functions are the same as the scalar functions $\cosh(-\theta) = \cosh\theta$, $\sinh(-\theta) = -\sinh\theta$, and $\tanh(-\theta) = -\tanh\theta$. As is seen in the proofs, this holds independently of the rotation axis as well as basis operator since these functions are a splitting of the relevant basis exponentials into odd and even components which then define the function properties with arguments of positive and negative signs. These relations for **coskk**, **sinkk**, and **tankk** are

$$\mathbf{coskk}(-\theta) = \mathbf{coskk}\theta$$

$$\mathbf{sinkk}(-\theta) \hat{k} \times \hat{k} \times = -\mathbf{sinkk}\theta \hat{k} \times \hat{k} \times$$

$$\text{nb: also } \mathbf{sinkk}(-\theta) = -\mathbf{sinkk}\theta$$

$$\mathbf{tankk}(-\theta) \hat{k} \times \hat{k} \times = -\mathbf{tankk}\theta \hat{k} \times \hat{k} \times$$

$$\text{nb: also } \mathbf{tankk}(-\theta) = -\mathbf{tankk}\theta$$

that is, corresponding to the hyperbolic scalars $\cosh(-\theta) = \cosh\theta$, $\sinh(-\theta) = -\sinh\theta$, and $\tanh(-\theta) = -\tanh\theta$

k -hat rotation axis

$$\mathbf{coskk}(-\theta) = \mathbf{coskk}\theta$$

j -hat rotation axis

$$\mathbf{cosjj}(-\theta) = \mathbf{cosjj}\theta$$

i -hat rotation axis

$$\mathbf{cosii}(-\theta) = \mathbf{cosii}\theta$$

$$\mathbf{sinkk}(-\theta) \hat{k} \times \hat{k} \times = -\mathbf{sinkk}\theta \hat{k} \times \hat{k} \times$$

$$\mathbf{sinjj}(-\theta) \hat{j} \times \hat{j} \times = -\mathbf{sinjj}\theta \hat{j} \times \hat{j} \times$$

$$\mathbf{sinii}(-\theta) \hat{i} \times \hat{i} \times = -\mathbf{sinii}\theta \hat{i} \times \hat{i} \times$$

$$\mathbf{tankk}(-\theta) \hat{k} \times \hat{k} \times = -\mathbf{tankk}\theta \hat{k} \times \hat{k} \times$$

$$\mathbf{tanjj}(-\theta) \hat{j} \times \hat{j} \times = -\mathbf{tanjj}\theta \hat{j} \times \hat{j} \times$$

$$\mathbf{tanii}(-\theta) \hat{i} \times \hat{i} \times = -\mathbf{tanii}\theta \hat{i} \times \hat{i} \times$$

The proofs in vector algebra and in diadically expanded matrix bases:

$$\mathbf{coskk}(-\theta) = \frac{1}{2} \left(e^{(-\theta)\hat{k} \times \hat{k} \times} + e^{-(\theta)\hat{k} \times \hat{k} \times} \right) = \frac{1}{2} \left(e^{-\theta\hat{k} \times \hat{k} \times} + e^{\theta\hat{k} \times \hat{k} \times} \right) = \frac{1}{2} \left(e^{\theta\hat{k} \times \hat{k} \times} + e^{-\theta\hat{k} \times \hat{k} \times} \right) = \mathbf{coskk}\theta$$

$$\mathbf{coskk}(-\theta) = \begin{pmatrix} \cosh(-\theta) & 0 & 0 \\ 0 & \cosh(-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{coskk}\theta$$

$$\mathbf{sinkk}(-\theta) \hat{k} \times \hat{k} \times = -\frac{1}{2} \left(e^{(-\theta)\hat{k} \times \hat{k} \times} - e^{-(\theta)\hat{k} \times \hat{k} \times} \right) = -\frac{1}{2} \left(e^{-\theta\hat{k} \times \hat{k} \times} - e^{\theta\hat{k} \times \hat{k} \times} \right) = \frac{1}{2} \left(e^{\theta\hat{k} \times \hat{k} \times} - e^{-\theta\hat{k} \times \hat{k} \times} \right) = -\mathbf{sinkk}\theta \hat{k} \times \hat{k} \times$$

$$\mathbf{sinkk}(-\theta) \hat{k} \times \hat{k} \times = \begin{pmatrix} -\sinh(-\theta) & 0 & 0 \\ 0 & -\sinh(-\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \sinh\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} -\sinh\theta & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\mathbf{sinkk}\theta \hat{k} \times \hat{k} \times$$

$$\mathbf{tankk}(-\theta) \hat{k} \times \hat{k} \times = (\mathbf{coskk}(-\theta))^{-1} \mathbf{sinkk}(-\theta) \hat{k} \times \hat{k} \times = (\mathbf{coskk}\theta)^{-1} \mathbf{sinkk}\theta \hat{k} \times \hat{k} \times = \mathbf{tankk}\theta \hat{k} \times \hat{k} \times$$

$$\mathbf{tankk}(-\theta) \hat{k} \times \hat{k} \times = \begin{pmatrix} -\tanh(-\theta) & 0 & 0 \\ 0 & -\tanh(-\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \tanh\theta & 0 & 0 \\ 0 & \tanh\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} -\tanh\theta & 0 & 0 \\ 0 & -\tanh\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\mathbf{tankk}\theta \hat{k} \times \hat{k} \times$$

The scalar properties of $\cosh(\theta + 2\pi ni) = \cosh\theta$, $\sinh(\theta + 2\pi ni) = \sinh\theta$, and $\tanh(\theta + 2\pi ni) = \tanh\theta$ are similar to those for $\mathbf{cosk}$, $\mathbf{sink}$, and $\mathbf{tank}$ and also hold for each coordinate axis. The difference being the $i = \hat{k} \times$ in the argument to return the double cross hyperbolic to the circular functions at the integers of 2π being $\hat{j} \times$ and $\hat{i} \times$ rather than $\hat{k} \times$. For $n = 0, \pm 1, \pm 2, \pm 3, \dots$ these are

$$\mathbf{coskk}(\theta + 2\pi n\hat{k} \times) = \mathbf{coskk}\theta$$

$$\mathbf{sinkk}(\theta + 2\pi n\hat{k} \times) \hat{k} \times \hat{k} \times = \mathbf{sinkk}\theta \hat{k} \times \hat{k} \times$$

$$\mathbf{tankk}(\theta + 2\pi n\hat{k} \times) \hat{k} \times \hat{k} \times = \mathbf{tankk}\theta \hat{k} \times \hat{k} \times$$

and, further, tabulating for each of the three principal axes

k -hat rotation axis

$$\mathbf{coskk}(\theta + 2\pi n\hat{k} \times) = \mathbf{coskk}\theta$$

j -hat rotation axis

$$\mathbf{cosjj}(\theta + 2\pi n\hat{j} \times) = \mathbf{cosjj}\theta$$

i -hat rotation axis

$$\mathbf{cosii}(\theta + 2\pi n\hat{i} \times) = \mathbf{cosii}\theta$$

$$\mathbf{sinkk}(\theta + 2\pi n\hat{k} \times) \hat{k} \times \hat{k} \times = \mathbf{sinkk}\theta \hat{k} \times \hat{k} \times$$

$$\mathbf{sinjj}(\theta + 2\pi n\hat{j} \times) \hat{j} \times \hat{j} \times = \mathbf{sinjj}\theta \hat{j} \times \hat{j} \times$$

$$\mathbf{sinii}(\theta + 2\pi n\hat{i} \times) \hat{i} \times \hat{i} \times = \mathbf{sinii}\theta \hat{i} \times \hat{i} \times$$

$$\mathbf{tankk}(\theta + 2\pi n\hat{k} \times) \hat{k} \times \hat{k} \times = \mathbf{tankk}\theta \hat{k} \times \hat{k} \times$$

$$\mathbf{tanjj}(\theta + 2\pi n\hat{j} \times) \hat{j} \times \hat{j} \times = \mathbf{tanjj}\theta \hat{j} \times \hat{j} \times$$

$$\mathbf{tanii}(\theta + 2\pi n\hat{i} \times) \hat{i} \times \hat{i} \times = \mathbf{tanii}\theta \hat{i} \times \hat{i} \times$$

Proof for the relation $\mathbf{coskk}(\theta + 2\pi n\hat{k} \times) = \mathbf{coskk}\theta$ for $\mathbf{coskk}$

$$\begin{aligned} \mathbf{coskk}(\theta + 2\pi n \hat{k} \times) &= \frac{1}{2} \left(e^{(\theta + 2\pi n \hat{k} \times) \hat{k} \times \hat{k} \times} + e^{-(\theta + 2\pi n \hat{k} \times) \hat{k} \times \hat{k} \times} \right) = \frac{1}{2} \left(e^{\theta \hat{k} \times \hat{k} \times} e^{-2\pi n \hat{k} \times} + e^{-\theta \hat{k} \times \hat{k} \times} e^{2\pi n \hat{k} \times} \right) = \frac{1}{2} \left(e^{\theta \hat{k} \times \hat{k} \times} + e^{-\theta \hat{k} \times \hat{k} \times} \right) \\ &= \mathbf{coskk}\theta \end{aligned}$$

since $e^{2\pi n (\hat{k} \times)^3} = e^{-2\pi n \hat{k} \times} = \mathbf{expk}(-2\pi n) = e^{-2\pi n (\hat{k} \times)^3} = e^{2\pi n \hat{k} \times} = \mathbf{expk}(-2\pi n) = I$ or in other words

$$\mathbf{exp}(\pm 2\pi n \hat{k} \times) = \mathbf{expk}(\pm 2\pi n) = \begin{pmatrix} \cos(2\pi n) & \mp \sin(2\pi n) & 0 \\ \pm \sin(2\pi n) & \cos(2\pi n) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

as diadically expanded matrix operators

$$\mathbf{coskk}(\theta + 2\pi n \hat{k} \times) = \begin{pmatrix} \cosh(\theta + 2\pi n \hat{k} \times) & 0 & 0 \\ 0 & \cosh(\theta + 2\pi n \hat{k} \times) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{coskk}\theta$$

Proof for the relation $\mathbf{sinkk}(\theta + 2\pi n \hat{k} \times) \hat{k} \times \hat{k} \times = \mathbf{sinkk}\theta \hat{k} \times \hat{k} \times$ for $\mathbf{sinkk}$

$$\begin{aligned} \mathbf{sinkk}(\theta + 2\pi n \hat{k} \times) \hat{k} \times \hat{k} \times &= -\frac{1}{2} \left(e^{(\theta + 2\pi n \hat{k} \times) \hat{k} \times \hat{k} \times} - e^{-(\theta + 2\pi n \hat{k} \times) \hat{k} \times \hat{k} \times} \right) = -\frac{1}{2} \left(e^{\theta \hat{k} \times \hat{k} \times} e^{-2\pi n \hat{k} \times} - e^{-\theta \hat{k} \times \hat{k} \times} e^{2\pi n \hat{k} \times} \right) = -\frac{1}{2} \left(e^{\theta \hat{k} \times \hat{k} \times} - e^{-\theta \hat{k} \times \hat{k} \times} \right) \\ &= \mathbf{sinkk}\theta \hat{k} \times \hat{k} \times \end{aligned}$$

as diadically expanded matrix operators

$$\mathbf{sinkk}(\theta + 2\pi n \hat{k} \times) \hat{k} \times \hat{k} \times = \begin{pmatrix} -\sinh(\theta + 2\pi n \hat{k} \times) & 0 & 0 \\ 0 & -\sinh(\theta + 2\pi n \hat{k} \times) & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\sinh\theta & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{sinkk}\theta \hat{k} \times \hat{k} \times$$

Proof similarly for $\mathbf{tankk}(\theta + 2\pi n \hat{k} \times) \hat{k} \times \hat{k} \times = \mathbf{tankk}\theta \hat{k} \times \hat{k} \times$

$$\mathbf{tankk}(\theta + 2\pi n \hat{k} \times) \hat{k} \times \hat{k} \times = \begin{pmatrix} -\tanh(\theta + 2\pi n \hat{k} \times) & 0 & 0 \\ 0 & -\tanh(\theta + 2\pi n \hat{k} \times) & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\tanh\theta & 0 & 0 \\ 0 & -\tanh\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{tankk}\theta \hat{k} \times \hat{k} \times$$

Proofs are similarity invariants for the hyperbolic periodicities for the other two principal axes (as well as the rotated polar cylindrical and spherical coordinate systems).

The hyperbolic relations for scalar hyperbolic sines, hyperbolic cosines, and hyperbolic tangents ($\sinh$, $\cosh$, and $\tanh$) are

(A) Hyperbolic Euler form

$$\cosh\theta + \sinh\theta = e^{\theta}$$

(B) Hyperbolic Conjugate Euler form

$$\cosh\theta - \sinh\theta = e^{-\theta}$$

(1) Basic hyperbolic relation for $\sinh$ and $\cosh$

$$\cosh^2\theta - \sinh^2\theta = 1$$

(2) Basic hyperbolic relation for $\tanh$ and sech

$$1 - \tanh^2\theta = \operatorname{sech}^2\theta$$

(3) Compound angle formula for $\sinh$

$$\sinh(\alpha + \beta) = \sinh\alpha \cosh\beta + \cosh\alpha \sinh\beta$$

(4) Compound angle formula for $\cosh$

$$\cosh(\alpha + \beta) = \cosh\alpha \cosh\beta + \sinh\alpha \sinh\beta$$

(5) Compound angle formula for $\tanh$

$$\tanh(\alpha + \beta) = (\tanh\alpha + \tanh\beta)(1 + \tanh\alpha \tanh\beta)^{-1}$$

(6) Double angle formula for $\sinh$

$$\sinh(2\alpha) = 2\sinh\alpha \cosh\alpha$$

(7) Double angle formula for $\cosh$

$$\cosh(2\alpha) = \cosh^2\alpha + \sinh^2\alpha$$

(8) Double angle formula for $\tanh$

$$\tanh(2\alpha) = 2\tanh\alpha(1 + \tanh^2\alpha)^{-1}$$

(9) Products of $\sinh$

$$2\sinh\alpha \sinh\beta = \cosh(\alpha + \beta) - \cosh(\alpha - \beta)$$

(10) Products of $\cosh$

$$2\cosh\alpha \cosh\beta = \cosh(\alpha + \beta) + \cosh(\alpha - \beta)$$

(11) Products of sinh and cosh

$$2\sinh\alpha\cosh\beta = \sinh(\alpha + \beta) + \sinh(\alpha - \beta)$$

The relations for xy plane ($\hat{k} \times$ rotation axis) basis hyperbolic sinh, cosh, and tanh (**sinkk**, **coskk**, and **tankk**) are

(A) Hyperbolic Euler form for **ciskk** = **expkk**

$$\coskk\theta - \text{sinkk}\theta \hat{k} \times \hat{k} \times = \expkk\theta$$

(B) Hyperbolic conjugate Euler form

$$\coskk\theta + \text{sinkk}\theta \hat{k} \times \hat{k} \times = \expkk(-\theta)$$

(1) Basic hyperbolic relation for **sinkk** and **coskk**

$$\coskk^2\theta - (\text{sinkk}\theta \hat{k} \times \hat{k} \times)^2 = 1$$

(2) Basic hyperbolic relation for **tankk** and **seckk**

$$1 - (\text{tankk}\theta \hat{k} \times \hat{k} \times)^2 = \text{seckk}^2\theta$$

(3) Compound angle formula for **sinkk**

$$\text{sinkk}(\alpha + \beta) \hat{k} \times \hat{k} \times = \coskk\alpha \text{sinkk}\beta \hat{k} \times \hat{k} \times + \coskk\beta \text{sinkk}\alpha \hat{k} \times \hat{k} \times$$

(4) Compound angle formula for **coskk**

$$\coskk(\alpha + \beta) = \coskk\alpha \coskk\beta + (\text{sinkk}\alpha \hat{k} \times \hat{k} \times)(\text{sinkk}\beta \hat{k} \times \hat{k} \times)$$

(5) Compound angle formula for **tankk**

$$\text{tankk}(\alpha + \beta) \hat{k} \times \hat{k} \times = \frac{\text{tankk}\alpha \hat{k} \times \hat{k} \times + \text{tankk}\beta \hat{k} \times \hat{k} \times}{1 + (\text{tankk}\alpha \hat{k} \times \hat{k} \times)(\text{tankk}\beta \hat{k} \times \hat{k} \times)}$$

(6) Double angle formula for **sinkk**

$$\text{sinkk}(2\alpha) \hat{k} \times \hat{k} \times = 2\coskk\alpha \text{sinkk}\alpha \hat{k} \times \hat{k} \times$$

(7) Double angle formula for **coskk**

$$\coskk(2\alpha) = \coskk^2\alpha + (\text{sinkk}\alpha \hat{k} \times \hat{k} \times)^2$$

(8) Double angle formula for **tankk**

$$\text{tankk}(2\alpha) \hat{k} \times \hat{k} \times = \left(2\text{tankk}\alpha \hat{k} \times \hat{k} \times\right) \left(1 + (\text{tankk}\alpha \hat{k} \times \hat{k} \times)^2\right)^{-1}$$

(9) Products of **sinkk**

$$2(\text{sinkk}\alpha \hat{k} \times \hat{k} \times)(\text{sinkk}\beta \hat{k} \times \hat{k} \times) = \coskk(\alpha + \beta) - \coskk(\alpha - \beta)$$

(10) Products of **coskk**

$$2\coskk\alpha \coskk\beta = \coskk(\alpha + \beta) + \coskk(\alpha - \beta)$$

(11) Products of **sinkk** and **coskk**

$$2(\text{sinkk}\alpha \hat{k} \times \hat{k} \times) \coskk\beta = \text{sinkk}(\alpha + \beta) \hat{k} \times \hat{k} \times + \text{sinkk}(\alpha - \beta) \hat{k} \times \hat{k} \times$$

Noting for reference that, for **sinkk** I have again coupled it with the double cross $\hat{k} \times \hat{k} \times$ (similarly as coupling **sink** with $\hat{k} \times$)

$$\text{sinkk}\theta = \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \sinh\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \quad \text{sinkk}\theta \hat{k} \times \hat{k} \times = \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \sinh\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\sinh\theta & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In complex number notation, this is essentially stating $\sinh\theta$ as $-i^2 \sinh\theta$ where $i^2 = \hat{k} \times \hat{k} \times$. Coupling the odd components with the basis operator became apparent was the best and necessary way to do things after power series expansion and the utility of the Euler forms of basis exponentials (which always couple the sine with i in cis).

Diadically expanded into matrix bases:

(A) Hyperbolic Euler form **ciskk** = **expkk**

$$\cosh\theta + \sinh\theta = e^\theta$$

$$\coskk\theta - \text{sinkk}\theta \hat{k} \times \hat{k} \times = \expkk\theta$$

$$\begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -\sinh\theta & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} e^\theta & 0 & 0 \\ 0 & e^\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(B) Hyperbolic conjugate Euler form

$$\cosh\theta - \sinh\theta = e^{-\theta}$$

$$\coskk\theta + \text{sinkk}\theta \hat{k} \times \hat{k} \times = \expkk(-\theta)$$

$$\begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -\sinh\theta & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(1) Basic hyperbolic relation for sinkk and coskk

$$\cosh^2 \theta - \sinh^2 \theta = 1 \qquad \text{coskk}^2 \theta - (\text{sinkk} \theta \hat{k} \times \hat{k} \times)^2 = 1$$

$$\begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} -\sinh \theta & 0 & 0 \\ 0 & -\sinh \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) Basic hyperbolic relation for tankk and seckk

$$1 - \tanh^2 \theta = \text{sech}^2 \theta \qquad 1 - (\text{tankk} \theta \hat{k} \times \hat{k} \times)^2 = \text{seckk}^2 \theta$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -\tanh \theta & 0 & 0 \\ 0 & -\tanh \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} \text{sech} \theta & 0 & 0 \\ 0 & \text{sech} \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2$$

(3) Compound angle formula for sinkk

$$\sinh(\alpha + \beta) = \sinh \alpha \cosh \beta + \cosh \alpha \sinh \beta \qquad \text{sinkk}(\alpha + \beta) \hat{k} \times \hat{k} \times = \text{coskk} \alpha \text{sinkk} \beta \hat{k} \times \hat{k} \times + \text{coskk} \beta \text{sinkk} \alpha \hat{k} \times \hat{k} \times$$

$$\begin{pmatrix} -\sinh(\alpha + \beta) & 0 & 0 \\ 0 & -\sinh(\alpha + \beta) & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sinh \beta & 0 & 0 \\ 0 & -\sinh \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & -\sinh \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(4) Compound angle formula for coskk

$$\cosh(\alpha + \beta) = \cosh \alpha \cosh \beta + \sinh \alpha \sinh \beta \qquad \text{coskk}(\alpha + \beta) = \text{coskk} \alpha \text{coskk} \beta + (\text{sinkk} \alpha \hat{k} \times \hat{k} \times)(\text{sinkk} \beta \hat{k} \times \hat{k} \times)$$

$$\begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & -\sinh \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sinh \beta & 0 & 0 \\ 0 & -\sinh \beta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(5) Compound angle formula for tankk

$$\tanh(\alpha + \beta) = \frac{\tanh \alpha + \tanh \beta}{1 + \tanh \alpha \tanh \beta} \qquad \text{tankk}(\alpha + \beta) \hat{k} \times \hat{k} \times = \frac{\text{tankk} \alpha \hat{k} \times \hat{k} \times + \text{tankk} \beta \hat{k} \times \hat{k} \times}{1 + (\text{tankk} \alpha \hat{k} \times \hat{k} \times)(\text{tankk} \beta \hat{k} \times \hat{k} \times)}$$

$$\begin{pmatrix} -\tanh(\alpha + \beta) & 0 & 0 \\ 0 & -\tanh(\alpha + \beta) & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\tanh \alpha - \tanh \beta & 0 & 0 \\ 0 & -\tanh \alpha - \tanh \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 + \tanh \alpha \tanh \beta & 0 & 0 \\ 0 & 1 + \tanh \alpha \tanh \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

(6) Double angle formula for sinkk

$$\sinh(2\alpha) = 2 \sinh \alpha \cosh \alpha \qquad \text{sinkk}(2\alpha) \hat{k} \times \hat{k} \times = 2 \text{coskk} \alpha \text{sinkk} \alpha \hat{k} \times \hat{k} \times$$

$$\begin{pmatrix} -\sinh 2\alpha & 0 & 0 \\ 0 & -\sinh 2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & -\sinh \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(7) Double angle formula for **coskk**

$$\cosh 2\alpha = \cosh^2 \alpha + \sinh^2 \alpha \quad \mathbf{coskk}(2\alpha) = \mathbf{coskk}^2 \alpha + (\mathbf{sinkk} \alpha \hat{k} \times \hat{k} \times)^2$$

$$\begin{pmatrix} \cosh 2\alpha & 0 & 0 \\ 0 & \cosh 2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 + \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & -\sinh \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}^2$$

(8) Double angle formula for **tankk**

$$\tanh(2\alpha) = 2 \tanh \alpha (1 + \tanh^2 \alpha)^{-1} \quad \mathbf{tankk}(2\alpha) \hat{k} \times \hat{k} \times = (2 \mathbf{tankk} \alpha \hat{k} \times \hat{k} \times) (1 + (\mathbf{tankk} \alpha \hat{k} \times \hat{k} \times)^2)^{-1}$$

$$\begin{pmatrix} -\tanh 2\alpha & 0 & 0 \\ 0 & -\tanh 2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} -\tanh \alpha & 0 & 0 \\ 0 & -\tanh \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 + \tanh^2 \alpha & 0 & 0 \\ 0 & 1 + \tanh^2 \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

(9) Products of **sink**

$$2 \sinh \alpha \sinh \beta = \cosh(\alpha + \beta) - \cosh(\alpha - \beta) \quad 2(\mathbf{sinkk} \alpha \hat{k} \times \hat{k} \times)(\mathbf{sinkk} \beta \hat{k} \times \hat{k} \times) = \mathbf{coskk}(\alpha + \beta) - \mathbf{coskk}(\alpha - \beta)$$

$$2 \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & -\sinh \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sinh \beta & 0 & 0 \\ 0 & -\sinh \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \cosh(\alpha - \beta) & 0 & 0 \\ 0 & \cosh(\alpha - \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(10) Products of **coskk**

$$2 \cosh \alpha \cosh \beta = \cosh(\alpha + \beta) + \cosh(\alpha - \beta) \quad 2 \mathbf{coskk} \alpha \mathbf{coskk} \beta = \mathbf{coskk}(\alpha + \beta) + \mathbf{coskk}(\alpha - \beta)$$

$$2 \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \cosh(\alpha - \beta) & 0 & 0 \\ 0 & \cosh(\alpha - \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(11) Products of **sinkk** and **coskk**

$$2 \sinh \alpha \cosh \beta = \sinh(\alpha + \beta) + \sinh(\alpha - \beta) \quad 2(\mathbf{sinkk} \alpha \hat{k} \times \hat{k} \times) \mathbf{coskk} \beta = \mathbf{sinkk}(\alpha + \beta) \hat{k} \times \hat{k} \times + \mathbf{sinkk}(\alpha - \beta) \hat{k} \times \hat{k} \times$$

$$2 \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & -\sinh \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\sinh(\alpha + \beta) & 0 & 0 \\ 0 & -\sinh(\alpha + \beta) & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\sinh(\alpha - \beta) & 0 & 0 \\ 0 & -\sinh(\alpha - \beta) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Over the next few pages are the proofs for firstly

(1) $\mathbf{coskk}^2 \theta - (\mathbf{sinkk} \theta \hat{k} \times \hat{k} \times)^2 = 1$ and then

(2) $1 - (\mathbf{tankk} \theta \hat{k} \times \hat{k} \times)^2 = \mathbf{seckk}^2 \theta$ and then

(3) $\mathbf{sinkk}(\alpha + \beta) \hat{k} \times \hat{k} \times = \mathbf{coskk} \alpha \mathbf{sinkk} \beta \hat{k} \times \hat{k} \times + \mathbf{coskk} \beta \mathbf{sinkk} \alpha \hat{k} \times \hat{k} \times$ and finally

(4) $\mathbf{coskk}(\alpha + \beta) = \mathbf{coskk} \alpha \mathbf{coskk} \beta + (\mathbf{sinkk} \alpha \hat{k} \times \hat{k} \times)(\mathbf{sinkk} \beta \hat{k} \times \hat{k} \times)$

from which each of the other relations can be obtained: half angle formulae by putting $\beta = \alpha$, and adding and subtracting the compound angle formulae for products.

(1) Proof of $\cos k k^2 \theta - (\sin k k \theta \hat{k} \times \hat{k} \times)^2 = 1$, the basic circular relation for **sinkk** and **coskk**

$$\begin{aligned} \cos k k \theta &= \frac{1}{2}(\exp k k(\theta) + \exp k k(-\theta)) & \sin k k \theta \hat{k} \times \hat{k} \times &= -\frac{1}{2}(\exp k k(\theta) - \exp k k(-\theta)) \\ &= \frac{1}{2}(e^{\theta \hat{k} \times \hat{k} \times} + e^{-\theta \hat{k} \times \hat{k} \times}) & &= -\frac{1}{2}(e^{\theta \hat{k} \times \hat{k} \times} - e^{-\theta \hat{k} \times \hat{k} \times}) \\ \cos k k^2 \theta - (\sin k k \theta \hat{k} \times \hat{k} \times)^2 &= \left(\frac{e^{\theta \hat{k} \times \hat{k} \times} + e^{-\theta \hat{k} \times \hat{k} \times}}{2} \right)^2 - \left(-\frac{e^{\theta \hat{k} \times \hat{k} \times} - e^{-\theta \hat{k} \times \hat{k} \times}}{2} \right)^2 \\ &= \frac{e^{2\theta \hat{k} \times \hat{k} \times} + 2 + e^{-2\theta \hat{k} \times \hat{k} \times}}{4} - \frac{e^{2\theta \hat{k} \times \hat{k} \times} - 2 + e^{-2\theta \hat{k} \times \hat{k} \times}}{4} \\ &= \frac{(e^{2\theta \hat{k} \times \hat{k} \times} - e^{2\theta \hat{k} \times \hat{k} \times}) + (2+2) + (e^{-2\theta \hat{k} \times \hat{k} \times} - e^{-2\theta \hat{k} \times \hat{k} \times})}{4} \\ &= 1 \end{aligned}$$

This relation holds for any basis function (here $\hat{k} \times \hat{k} \times$ and also $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$) since it is a property of the separation of the odd and even components of the basis exponential of that basis function and any basis function commutes with itself and its conjugate (and scalars).

Similarly as for the circular relation in the quartic root of the k -hat axis, $\hat{k} \times$, I change the base of the hyperbolic relation in the double cross $\hat{k} \times \hat{k} \times$ to j -hat and i -hat just to show that it all works out. Similarly as for $\hat{k} \times$, all of these for $\hat{k} \times \hat{k} \times$ and change of base to $\hat{j} \times \hat{j} \times$ and $\hat{i} \times \hat{i} \times$ also work similarly for r -hat, θ -hat, and ϕ -hat in polar (rotated) systems (and so also for $\hat{r} \times \hat{r} \times$, $\hat{\theta} \times \hat{\theta} \times$, and $\hat{\phi} \times \hat{\phi} \times$).

In terms of diadically expanded matrix operators with

$$\begin{aligned} \cos k k \theta &= \frac{1}{2}(\exp k k(\theta) + \exp k k(-\theta)) & \sin k k \theta \hat{k} \times \hat{k} \times &= -\frac{1}{2}(\exp k k(\theta) - \exp k k(-\theta)) \\ &= \begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} & &= \begin{pmatrix} -\sinh \theta & 0 & 0 \\ 0 & -\sinh \theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

the proof is

$$\begin{aligned} \begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} -\sinh \theta & 0 & 0 \\ 0 & -\sinh \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 &= \begin{pmatrix} \cosh^2 \theta & 0 & 0 \\ 0 & \cosh^2 \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \sinh^2 \theta & 0 & 0 \\ 0 & \sinh^2 \theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cosh^2 \theta - \sinh^2 \theta & 0 & 0 \\ 0 & \cosh^2 \theta - \sinh^2 \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= I \end{aligned}$$

Note that next changes of base are the same changes of base as for the circular relation in **cosk** and **sink**. and it is also the same change of base for the axis osculating hyperbolic relation in **coshk** and **sinhk** which I have chosen to omit.

k -hat to j -hat principal rotation axis (xy plane to zx plane): **making a change of base** from the k -hat principal rotation axis sign osculating hyperbolic relation in **coskk** and **sinkk** to the j -hat principal rotation axis for the congruent relation in **cosjj** and **sinjj** by applying the full rank complex number for the quarter rotation around i -hat **expi**($\pi/2$)

$$1 = \cos k k^2 \theta - (\sin k k \theta \hat{k} \times \hat{k} \times)^2$$

hyperbolic relation for $\hat{k}$ axis

$$\exp i(\pi/2)(1)\exp i(-\pi/2) = \exp i(\pi/2) \left(\cos k k^2 \theta - (\sin k k \theta \hat{k} \times \hat{k} \times)^2 \right) \exp i(-\pi/2)$$

change of base

$$\exp i(\pi/2)\exp i(-\pi/2) = \exp i(\pi/2) \left(\cos k k^2 \theta \right) \exp i(-\pi/2) - \exp i(\pi/2) \left((\sin k k \theta \hat{k} \times \hat{k} \times)^2 \right) \exp i(-\pi/2)$$

$$1 = \cos j j^2 \theta - (\sin j j \theta \hat{j} \times \hat{j} \times)^2$$

hyperbolic relation for $\hat{j}$ axis

exactly for the circular relations, with change of base operator $\exp i$ in diadically expanded matrix form $\exp i(\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\exp i(\pi/2)(1)\exp i(-\pi/2) = \exp i(\pi/2) \left(\cos k k^2 \theta - (\sin k k \theta \hat{k} \times \hat{k} \times)^2 \right) \exp i(-\pi/2)$$

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} I \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \left(\begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} -\sinh \theta & 0 & 0 \\ 0 & -\sinh \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ I &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \left(\begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\sinh \theta & 0 & 0 \\ 0 & -\sinh \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \theta \end{pmatrix}^2 - \begin{pmatrix} -\sinh \theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \theta \end{pmatrix}^2 \\ &= \cos j j^2 \theta - (\sin j j \theta \hat{j} \times \hat{j} \times)^2 \end{aligned}$$

which is the sign osculating hyperbolic relation for the zx plane, $\cos j j^2 \theta - (\sin j j \theta \hat{j} \times \hat{j} \times)^2 = 1$, after the change of base from the k -hat rotation axis to that of the j -hat rotation axis.

k -hat to i -hat principal rotation axis (xy plane to yz plane): **making a change of base** from the k -hat principal rotation axis axis sign osculating hyperbolic relation in $\cos k k$ and $\sin k k$ to the i -hat principal rotation axis for the congruent relation in $\cos i i$ and $\sin i i$ by applying the full rank complex number for the antisense (transpose) quarter rotation around j -hat $\exp j(-\pi/2)$

$$1 = \cos k k^2 \theta - (\sin k k \theta \hat{k} \times \hat{k} \times)^2$$

hyperbolic relation for $\hat{k}$ axis

$$\exp j(-\pi/2)(1)\exp j(\pi/2) = \exp j(-\pi/2) \left(\cos k k^2 \theta - (\sin k k \theta \hat{k} \times \hat{k} \times)^2 \right) \exp j(\pi/2)$$

change of base

$$\exp j(-\pi/2)\exp j(\pi/2) = \exp j(-\pi/2) \left(\cos k k^2 \theta \right) \exp j(\pi/2) - \exp j(-\pi/2) \left((\sin k k \theta \hat{k} \times \hat{k} \times)^2 \right) \exp j(\pi/2)$$

$$1 = \cos i i^2 \theta - (\sin i i \theta \hat{i} \times \hat{i} \times)^2$$

hyperbolic relation for $\hat{j}$ axis

and with change of base operator $\exp j$ in diadically expanded matrix form $\exp j(-\pi/2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$\exp j(-\pi/2)(1)\exp j(\pi/2) = \exp j(-\pi/2) \left(\cos k k^2 \theta - (\sin k k \theta \hat{k} \times \hat{k} \times)^2 \right) \exp j(\pi/2)$$

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} I \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \left(\begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} -\sinh \theta & 0 & 0 \\ 0 & -\sinh \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 \right) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned}
I &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sinh \theta & 0 & 0 \\ 0 & -\sinh \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & \cosh \theta \end{pmatrix}^2 - \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh \theta & 0 \\ 0 & 0 & -\sinh \theta \end{pmatrix}^2 \\
&= \mathbf{cosii}^2 \theta - (\mathbf{sinii} \theta \hat{i} \times \hat{i} \times)^2
\end{aligned}$$

which is the basic sign osculating double cross hyperbolic relation for the yz plane, $\mathbf{cosii}^2 \theta - (\mathbf{sinii} \theta \hat{i} \times \hat{i} \times)^2 = 1$, after the change of base from the $\hat{k}$ -hat rotation axis (with xy plane orthogonal complement) to that of the $\hat{i}$ -hat rotation axis (with yz plane orthogonal complement).

To show that the double cross hyperbolic relation $\mathbf{coskk}^2 \theta - (\mathbf{sinkk} \theta \hat{k} \times \hat{k} \times)^2 = 1$ the xy-plane holds good similarly as the circular relation after the change of base to the $\hat{j}$ -hat and $\hat{i}$ -hat axes, here are the proofs for $\mathbf{cosjj}^2 \theta - (\mathbf{sinjj} \theta \hat{j} \times \hat{j} \times)^2 = 1$ for the zx-plane and $\mathbf{cosii}^2 \theta - (\mathbf{sinii} \theta \hat{i} \times \hat{i} \times)^2 = 1$ for the yz-plane to show that these relations are equal to changing base to these principal axes from $\mathbf{coskk}^2 \theta - (\mathbf{sinkk} \theta \hat{k} \times \hat{k} \times)^2 = 1$:

Proof of $\mathbf{cosjj}^2 \theta - (\mathbf{sinjj} \theta \hat{j} \times \hat{j} \times)^2 = 1$, the double cross hyperbolic relation for $\mathbf{sinjj}$ and $\mathbf{cosjj}$

$$\begin{aligned}
\mathbf{cosjj} \theta &= \frac{1}{2}(\mathbf{expjj}(\theta) + \mathbf{expjj}(-\theta)) & \mathbf{sinjj} \theta \hat{j} \times \hat{j} \times &= -\frac{1}{2}(\mathbf{expjj}(\theta) - \mathbf{expjj}(-\theta)) \\
&= \frac{1}{2}(e^{\theta \hat{j} \times \hat{j} \times} + e^{-\theta \hat{j} \times \hat{j} \times}) & &= -\frac{1}{2}(e^{\theta \hat{j} \times \hat{j} \times} - e^{-\theta \hat{j} \times \hat{j} \times}) \\
\mathbf{cosjj}^2 \theta - (\mathbf{sinjj} \theta \hat{j} \times \hat{j} \times)^2 &= \left(\frac{e^{\theta \hat{j} \times \hat{j} \times} + e^{-\theta \hat{j} \times \hat{j} \times}}{2} \right)^2 - \left(-\frac{e^{\theta \hat{j} \times \hat{j} \times} - e^{-\theta \hat{j} \times \hat{j} \times}}{2} \right)^2 \\
&= \left(\frac{e^{\theta \hat{j} \times \hat{j} \times} + e^{-\theta \hat{j} \times \hat{j} \times}}{2} + \frac{e^{\theta \hat{j} \times \hat{j} \times} - e^{-\theta \hat{j} \times \hat{j} \times}}{2} \right) \left(\frac{e^{\theta \hat{j} \times \hat{j} \times} + e^{-\theta \hat{j} \times \hat{j} \times}}{2} - \frac{e^{\theta \hat{j} \times \hat{j} \times} - e^{-\theta \hat{j} \times \hat{j} \times}}{2} \right) \\
&= e^{\theta \hat{j} \times \hat{j} \times} e^{-\theta \hat{j} \times \hat{j} \times} \\
&= 1
\end{aligned}$$

which is complex algebra in $i^2 = (\hat{j} \times)^2$ for the zx plane (rather than xy plane for $i^2 = (\hat{k} \times)^2$) since it is for a single rotation axis so that $\hat{j} \times$ commutes with itself and its conjugate (and other scalars) on this plane and is the same as that from a change of base from $\hat{k} \times$ to $\hat{j} \times$. The diadically expanded matrix basis operators for the $\hat{j}$ -hat double cross hyperbolic cosine and sine are

$$\begin{aligned}
\mathbf{cosjj} \theta &= \frac{1}{2}(\mathbf{expjj}(\theta) + \mathbf{expjj}(-\theta)) & \mathbf{sinjj} \theta \hat{j} \times \hat{j} \times &= -\frac{1}{2}(\mathbf{expjj}(\theta) - \mathbf{expjj}(-\theta)) \\
&= \begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \theta \end{pmatrix} & &= \begin{pmatrix} -\sinh \theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \theta \end{pmatrix}
\end{aligned}$$

so the proof in terms of the diadically expanded matrix base upon substituting into $\mathbf{cosjj}^2 \theta - (\mathbf{sinjj} \theta \hat{j} \times \hat{j} \times)^2$ is

$$\begin{aligned}
&\mathbf{cosjj}^2 \theta - (\mathbf{sinjj} \theta \hat{j} \times \hat{j} \times)^2 \\
&= \begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \theta \end{pmatrix}^2 - \begin{pmatrix} -\sinh \theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \theta \end{pmatrix}^2 = \begin{pmatrix} \cosh^2 \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh^2 \theta \end{pmatrix} - \begin{pmatrix} \sinh^2 \theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sinh^2 \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= I
\end{aligned}$$

For the i -hat principal axis and proof of $\mathbf{cosii}^2\theta - (\mathbf{sinii}\theta\hat{i} \times \hat{i} \times)^2 = 1$, the basic double cross hyperbolic relations for $\mathbf{sinii}$ and $\mathbf{cosii}$ are

$$\begin{aligned}\mathbf{cosii} \theta &= \frac{1}{2}(\mathbf{expii}(\theta) + \mathbf{expii}(-\theta)) & \mathbf{sinii} \theta \hat{i} \times \hat{i} \times &= -\frac{1}{2}(\mathbf{expii}(\theta) - \mathbf{expii}(-\theta)) \\ &= \frac{1}{2}(e^{\theta\hat{i} \times \hat{i} \times} + e^{-\theta\hat{i} \times \hat{i} \times}) & &= -\frac{1}{2}(e^{\theta\hat{i} \times \hat{i} \times} - e^{-\theta\hat{i} \times \hat{i} \times})\end{aligned}$$

so

$$\begin{aligned}\mathbf{cosii}^2\theta - (\mathbf{sinii}\theta\hat{i} \times \hat{i} \times)^2 &= \left(\frac{e^{\theta\hat{i} \times \hat{i} \times} + e^{-\theta\hat{i} \times \hat{i} \times}}{2}\right)^2 - \left(-\frac{e^{\theta\hat{i} \times \hat{i} \times} - e^{-\theta\hat{i} \times \hat{i} \times}}{2}\right)^2 \\ &= \left(\frac{e^{\theta\hat{i} \times \hat{i} \times} + e^{-\theta\hat{i} \times \hat{i} \times}}{2} + \frac{e^{\theta\hat{i} \times \hat{i} \times} - e^{-\theta\hat{i} \times \hat{i} \times}}{2}\right) \left(\frac{e^{\theta\hat{i} \times \hat{i} \times} + e^{-\theta\hat{i} \times \hat{i} \times}}{2} - \frac{e^{\theta\hat{i} \times \hat{i} \times} - e^{-\theta\hat{i} \times \hat{i} \times}}{2}\right) \\ &= e^{\theta\hat{i} \times \hat{i} \times} e^{-\theta\hat{i} \times \hat{i} \times} \\ &= 1\end{aligned}$$

which is complex algebra in $i^2 = (\hat{i} \times)^2$ for the yz plane (rather than xy plane for $i^2 = (\hat{k} \times)^2$ or the xz plane for $i^2 = (\hat{j} \times)^2$) since it is for a basis operator so that $\hat{i} \times \hat{i} \times$ commutes with itself and its conjugate (and other scalars)

$$\begin{aligned}\mathbf{cosii} \theta &= \frac{1}{2}(\mathbf{expii}(\theta) + \mathbf{expii}(-\theta)) & \mathbf{sinii} \theta \hat{i} \times \hat{i} \times &= -\frac{1}{2}(\mathbf{expii}(\theta) - \mathbf{expii}(-\theta)) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix} & &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & -\sinh\theta \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{cosii}^2\theta - (\mathbf{sinii}\theta\hat{i} \times \hat{i} \times)^2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix}^2 - \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & -\sinh\theta \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh^2\theta & 0 \\ 0 & 0 & \cosh^2\theta \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh^2\theta & 0 \\ 0 & 0 & -\sinh^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= I\end{aligned}$$

so it all works as it should.

(2) Similarly as the circular relation in **tank**, $1 - (\mathbf{tank} \theta \hat{k} \times)^2 = \mathbf{seck}^2\theta$, the double cross (as well as the axis osculating) hyperbolic relation for **tank**, $1 - (\mathbf{tankk} \theta \hat{k} \times \hat{k} \times)^2 = \mathbf{seckk}^2\theta$, is obtained by multiplying both sides of $\mathbf{coskk}^2\theta - (\mathbf{sinkk} \theta \hat{k} \times \hat{k} \times)^2 = 1$ by $\mathbf{seckk}^2\theta$

$$\begin{aligned}\mathbf{seckk}^2\theta (\mathbf{coskk}^2\theta - (\mathbf{sinkk} \theta \hat{k} \times \hat{k} \times)^2) &= \mathbf{seckk}^2\theta \\ 1 - (\mathbf{tankk} \theta \hat{k} \times \hat{k} \times)^2 &= \mathbf{seckk}^2\theta\end{aligned}$$

where $\mathbf{seckk}^2\theta \mathbf{coskk}^2\theta = 1$ and $\mathbf{seckk}^2\theta (\mathbf{sinkk} \theta \hat{k} \times \hat{k} \times)^2 = (\mathbf{tankk} \theta \hat{k} \times \hat{k} \times)^2$

$$\text{that is, from } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} -\sinh\theta & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} \cosh^2\theta & 0 & 0 \\ 0 & \cosh^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \sinh^2\theta & 0 & 0 \\ 0 & \sinh^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{multiplying both sides by } \mathbf{coskk}^{-2}\theta = \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-2} = \begin{pmatrix} \cosh^{-2}\theta & 0 & 0 \\ 0 & \cosh^{-2}\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \text{sech}^2\theta & 0 & 0 \\ 0 & \text{sech}^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{seckk}^2\theta$$

the $\text{cosk}\theta - (\text{sink}\theta \hat{k} \times \hat{k} \times)^2 = 1$ relation converts to $1 - (\text{tank}\theta \hat{k} \times \hat{k} \times)^2 = \text{seck}^2\theta$ by multiplication through by $\text{seck}^2\theta$

$$\begin{aligned} \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-2} &= \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-2} \left(\begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} -\sinh\theta & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 \right) \\ \begin{pmatrix} \text{sech}^2\theta & 0 & 0 \\ 0 & \text{sech}^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} \text{sech}^2\theta & 0 & 0 \\ 0 & \text{sech}^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh^2\theta & 0 & 0 \\ 0 & \cosh^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \text{sech}^2\theta & 0 & 0 \\ 0 & \text{sech}^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sinh^2\theta & 0 & 0 \\ 0 & \sinh^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} \text{sech}^2\theta & 0 & 0 \\ 0 & \text{sech}^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \tanh^2\theta & 0 & 0 \\ 0 & \tanh^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \text{seck}^2\theta &= 1 - (\text{tank}\theta \hat{k} \times \hat{k} \times)^2 \end{aligned}$$

which is the required relation.

(3) Proof of $\text{sinkk}(\alpha + \beta) \hat{k} \times \hat{k} \times = \text{cosk}\alpha \text{sink}\beta \hat{k} \times \hat{k} \times + \text{cosk}\beta \text{sink}\alpha \hat{k} \times \hat{k} \times$, the compound argument formula for sinh

$$\begin{aligned} \text{cosk}\alpha \text{sink}\beta \hat{k} \times \hat{k} \times + \text{cosk}\beta \text{sink}\alpha \hat{k} \times \hat{k} \times &= \left(\frac{e^{\alpha \hat{k} \times \hat{k} \times} + e^{-\alpha \hat{k} \times \hat{k} \times}}{2} \right) \left(\frac{e^{\beta \hat{k} \times \hat{k} \times} - e^{-\beta \hat{k} \times \hat{k} \times}}{2} \right) \\ &+ \left(-\frac{e^{\beta \hat{k} \times \hat{k} \times} + e^{-\beta \hat{k} \times \hat{k} \times}}{2} \right) \left(-\frac{e^{\alpha \hat{k} \times \hat{k} \times} - e^{-\alpha \hat{k} \times \hat{k} \times}}{2} \right) \\ &= \frac{1}{4} \left(e^{\alpha \hat{k} \times \hat{k} \times} e^{\beta \hat{k} \times \hat{k} \times} - e^{\alpha \hat{k} \times \hat{k} \times} e^{-\beta \hat{k} \times \hat{k} \times} + e^{-\alpha \hat{k} \times \hat{k} \times} e^{\beta \hat{k} \times \hat{k} \times} - e^{-\alpha \hat{k} \times \hat{k} \times} e^{-\beta \hat{k} \times \hat{k} \times} \right) \\ &+ \frac{1}{4} \left(e^{\beta \hat{k} \times \hat{k} \times} e^{\alpha \hat{k} \times \hat{k} \times} - e^{\beta \hat{k} \times \hat{k} \times} e^{-\alpha \hat{k} \times \hat{k} \times} + e^{-\beta \hat{k} \times \hat{k} \times} e^{\alpha \hat{k} \times \hat{k} \times} - e^{-\beta \hat{k} \times \hat{k} \times} e^{-\alpha \hat{k} \times \hat{k} \times} \right) \\ &= \frac{1}{4} \left(e^{\alpha \hat{k} \times \hat{k} \times} e^{\beta \hat{k} \times \hat{k} \times} - e^{-\alpha \hat{k} \times \hat{k} \times} e^{-\beta \hat{k} \times \hat{k} \times} \right) \\ &= \frac{1}{4} \left(e^{(\alpha + \beta) \hat{k} \times \hat{k} \times} - e^{-(\alpha + \beta) \hat{k} \times \hat{k} \times} \right) \\ &= \text{sinkk}(\alpha + \beta) \hat{k} \times \hat{k} \times \end{aligned}$$

where $e^{\alpha \hat{k} \times \hat{k} \times}$ and $e^{\beta \hat{k} \times \hat{k} \times}$ and their conjugates commute since they are basis exponentials for the same basis foci and basis operator $\hat{k} \times \hat{k} \times$. In diadically expanded matrix operators

$$\begin{aligned} \text{cosk}\alpha \text{sink}\beta \hat{k} \times \hat{k} \times + \text{cosk}\beta \text{sink}\alpha \hat{k} \times \hat{k} \times &= \begin{pmatrix} \cosh\alpha & 0 & 0 \\ 0 & \cosh\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sinh\beta & 0 & 0 \\ 0 & -\sinh\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cosh\beta & 0 & 0 \\ 0 & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sinh\alpha & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\cosh\alpha \sinh\beta & 0 & 0 \\ 0 & -\cosh\alpha \sinh\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\cosh\beta \sinh\alpha & 0 & 0 \\ 0 & -\cosh\beta \sinh\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sinh(\alpha + \beta) & 0 & 0 \\ 0 & -\sinh(\alpha + \beta) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \text{sinkk}(\alpha + \beta) \hat{k} \times \hat{k} \times \end{aligned}$$

Changing base to the other principal rotation axes:

$\hat{k}$ -hat to $\hat{j}$ -hat principal rotation axis (xy plane to zx plane): changing base to the $\hat{j}$ -hat rotation axes with $\text{expi}(\pi/2)$ and $\text{expi}(-\pi/2)$

$$\begin{aligned}
\text{sinkk}(\alpha + \beta) \hat{k} \times \hat{k} \times &= \text{coskk}\alpha \text{sinkk}\beta \hat{k} \times \hat{k} \times + \text{coskk}\beta \text{sinkk}\alpha \hat{k} \times \hat{k} \times && \text{compound angle formula for sinkk} \\
\text{expi}(\pi/2) \left(\text{sinkk}(\alpha + \beta) \hat{k} \times \hat{k} \times \right) \text{expi}(-\pi/2) &= \text{expi}(\pi/2) \left(\text{coskk}\alpha \text{sinkk}\beta \hat{k} \times \hat{k} \times + \text{coskk}\beta \text{sinkk}\alpha \hat{k} \times \hat{k} \times \right) \text{expi}(-\pi/2) && \text{change of base} \\
\text{sinjj}(\alpha + \beta) \hat{j} \times \hat{j} \times &= \text{expi}(\pi/2) \left(\text{coskk}\alpha \text{sinkk}\beta \hat{k} \times \hat{k} \times \right) \text{expi}(-\pi/2) \\
&\quad + \text{expi}(\pi/2) \left(\text{coskk}\beta \text{sinkk}\alpha \hat{k} \times \hat{k} \times \right) \text{expi}(-\pi/2) \\
&= \text{expi}(\pi/2) \left(\text{coskk}\alpha \right) \text{expi}(-\pi/2) \text{expi}(\pi/2) \left(\text{sinkk}\beta \hat{k} \times \hat{k} \times \right) \text{expi}(-\pi/2) \\
&\quad + \text{expi}(\pi/2) \left(\text{coskk}\beta \right) \text{expi}(-\pi/2) \text{expi}(\pi/2) \left(\text{sinkk}\alpha \hat{k} \times \hat{k} \times \right) \text{expi}(-\pi/2) \\
\text{sinjj}(\alpha + \beta) \hat{j} \times \hat{j} \times &= \text{cosjj}\alpha \text{sinjj}\beta \hat{j} \times \hat{j} \times + \text{cosjj}\beta \text{sinjj}\alpha \hat{j} \times \hat{j} \times && \text{compound angle formula for sinjj}
\end{aligned}$$

In matrix operators

$$\begin{aligned}
\text{expi}(\pi/2) \left(\text{sinkk}(\alpha + \beta) \hat{k} \times \hat{k} \times \right) \text{expi}(-\pi/2) &= \text{expi}(\pi/2) \left(\text{coskk}\alpha \text{sinkk}\beta \hat{k} \times \hat{k} \times + \text{coskk}\beta \text{sinkk}\alpha \hat{k} \times \hat{k} \times \right) \text{expi}(-\pi/2) \\
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\sinh(\alpha + \beta) & 0 & 0 \\ 0 & -\sinh(\alpha + \beta) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cosh\alpha & 0 & 0 \\ 0 & \cosh\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sinh\beta & 0 & 0 \\ 0 & -\sinh\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cosh\beta & 0 & 0 \\ 0 & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sinh\alpha & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
\begin{pmatrix} -\sinh(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh(\alpha + \beta) \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cosh\alpha & 0 & 0 \\ 0 & \cosh\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\sinh\beta & 0 & 0 \\ 0 & -\sinh\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cosh\beta & 0 & 0 \\ 0 & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\sinh\alpha & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
\begin{pmatrix} -\sinh(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh(\alpha + \beta) \end{pmatrix} &= \begin{pmatrix} \cosh\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\alpha \end{pmatrix} \begin{pmatrix} -\sinh\beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh\beta \end{pmatrix} + \begin{pmatrix} \cosh\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\beta \end{pmatrix} \begin{pmatrix} -\sinh\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh\alpha \end{pmatrix} \\
\text{sinjj}(\alpha + \beta) \hat{j} \times \hat{j} \times &= \text{cosjj}\alpha \text{sinjj}\beta \hat{j} \times \hat{j} \times + \text{cosjj}\beta \text{sinjj}\alpha \hat{j} \times \hat{j} \times
\end{aligned}$$

which is the required relation for the sinh compound argument formula for the $\hat{j}$ -hat rotation axis (for **sinjj**) base changed from the $\hat{k}$ -hat rotation axis (for **sinkk**).

$\hat{k}$ -hat to $\hat{i}$ -hat principal rotation axis (xy plane to yz plane): changing base to the $\hat{i}$ -hat principal rotation axes with **expj**($-\pi/2$) and **expj**($\pi/2$)

$$\begin{aligned}
\text{sinkk}(\alpha + \beta) \hat{k} \times \hat{k} \times &= \text{coskk}\alpha \text{sinkk}\beta \hat{k} \times \hat{k} \times + \text{coskk}\beta \text{sinkk}\alpha \hat{k} \times \hat{k} \times && \text{compound angle formula for sinkk} \\
\text{expj}(-\pi/2) \left(\text{sinkk}(\alpha + \beta) \hat{k} \times \hat{k} \times \right) \text{expj}(\pi/2) &= \text{expj}(-\pi/2) \left(\text{coskk}\alpha \text{sinkk}\beta \hat{k} \times \hat{k} \times + \text{coskk}\beta \text{sinkk}\alpha \hat{k} \times \hat{k} \times \right) \text{expj}(\pi/2) && \text{change of base} \\
\text{sinii}(\alpha + \beta) \hat{i} \times \hat{i} \times &= \text{expj}(-\pi/2) \left(\text{coskk}\alpha \text{sinkk}\beta \hat{k} \times \hat{k} \times \right) \text{expj}(\pi/2) \\
&\quad + \text{expj}(-\pi/2) \left(\text{coskk}\beta \text{sinkk}\alpha \hat{k} \times \hat{k} \times \right) \text{expj}(\pi/2) \\
&= \text{expj}(-\pi/2) \left(\text{coskk}\alpha \right) \text{expj}(\pi/2) \text{expj}(-\pi/2) \left(\text{sinkk}\beta \hat{k} \times \hat{k} \times \right) \text{expj}(\pi/2) \\
&\quad + \text{expj}(-\pi/2) \left(\text{coskk}\beta \right) \text{expj}(\pi/2) \text{expj}(-\pi/2) \left(\text{sinkk}\alpha \hat{k} \times \hat{k} \times \right) \text{expj}(\pi/2) \\
\text{sinii}(\alpha + \beta) \hat{i} \times \hat{i} \times &= \text{cosii}\alpha \text{sinii}\beta \hat{i} \times \hat{i} \times + \text{cosii}\beta \text{sinii}\alpha \hat{i} \times \hat{i} \times && \text{compound angle formula for sinii}
\end{aligned}$$

In matrix operators

$$\begin{aligned}
& \expj(-\pi/2) \left(\text{sinkk}(\alpha + \beta) \hat{k} \times \hat{k} \times \right) \expj(\pi/2) = \expj(-\pi/2) \left(\text{coskk}\alpha \text{sinkk}\beta \hat{k} \times \hat{k} \times + \text{coskk}\beta \text{sinkk}\alpha \hat{k} \times \hat{k} \times \right) \expj(\pi/2) \\
& \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sinh(\alpha + \beta) & 0 & 0 \\ 0 & -\sinh(\alpha + \beta) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh\alpha & 0 & 0 \\ 0 & \cosh\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sinh\beta & 0 & 0 \\ 0 & -\sinh\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
& + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh\beta & 0 & 0 \\ 0 & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sinh\alpha & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
& \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh(\alpha + \beta) & 0 \\ 0 & 0 & -\sinh(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh\alpha & 0 & 0 \\ 0 & \cosh\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sinh\beta & 0 & 0 \\ 0 & -\sinh\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
& + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh\beta & 0 & 0 \\ 0 & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sinh\alpha & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
& \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh(\alpha + \beta) & 0 \\ 0 & 0 & -\sinh(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\alpha & 0 \\ 0 & 0 & \cosh\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\beta & 0 \\ 0 & 0 & -\sinh\beta \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\beta & 0 \\ 0 & 0 & \cosh\beta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & -\sinh\alpha \end{pmatrix} \\
& \text{sinii}(\alpha + \beta) \hat{i} \times \hat{i} \times = \text{cosii}\alpha \text{sinii}\beta \hat{i} \times \hat{i} \times + \text{cosii}\beta \text{sinii}\alpha \hat{i} \times \hat{i} \times
\end{aligned}$$

which is the required relation for the sinh compound argument formula for the $\hat{i}$ -hat rotation axis (for **sinii**) base changed from the $\hat{k}$ -hat rotation axis (from **sinkk**).

(4) Proof of $\text{coskk}(\alpha + \beta) = \text{coskk}\alpha \text{coskk}\beta + (\text{sinkk}\alpha \hat{k} \times \hat{k} \times)(\text{sinkk}\beta \hat{k} \times \hat{k} \times)$, the compound angle formula for cosine

$$\begin{aligned}
& \text{coskk}\alpha \text{coskk}\beta + (\text{sinkk}\alpha \hat{k} \times \hat{k} \times)(\text{sinkk}\beta \hat{k} \times \hat{k} \times) = \frac{1}{2} \left(e^{\alpha \hat{k} \times \hat{k} \times} + e^{-\alpha \hat{k} \times \hat{k} \times} \right) \frac{1}{2} \left(e^{\beta \hat{k} \times \hat{k} \times} + e^{-\beta \hat{k} \times \hat{k} \times} \right) \\
& + \frac{1}{2} \left(e^{\alpha \hat{k} \times \hat{k} \times} - e^{-\alpha \hat{k} \times \hat{k} \times} \right) \frac{1}{2} \left(e^{\beta \hat{k} \times \hat{k} \times} - e^{-\beta \hat{k} \times \hat{k} \times} \right) \\
& = \frac{1}{4} \left(e^{\alpha \hat{k} \times \hat{k} \times} e^{\beta \hat{k} \times \hat{k} \times} + e^{\alpha \hat{k} \times \hat{k} \times} e^{-\beta \hat{k} \times \hat{k} \times} + e^{-\alpha \hat{k} \times \hat{k} \times} e^{\beta \hat{k} \times \hat{k} \times} + e^{-\alpha \hat{k} \times \hat{k} \times} e^{-\beta \hat{k} \times \hat{k} \times} \right) \\
& + \frac{1}{4} \left(e^{\alpha \hat{k} \times \hat{k} \times} e^{\beta \hat{k} \times \hat{k} \times} - e^{\alpha \hat{k} \times \hat{k} \times} e^{-\beta \hat{k} \times \hat{k} \times} - e^{-\alpha \hat{k} \times \hat{k} \times} e^{\beta \hat{k} \times \hat{k} \times} + e^{-\alpha \hat{k} \times \hat{k} \times} e^{-\beta \hat{k} \times \hat{k} \times} \right) \\
& = \frac{1}{2} \left(e^{\alpha \hat{k} \times \hat{k} \times} e^{\beta \hat{k} \times \hat{k} \times} + e^{-\alpha \hat{k} \times \hat{k} \times} e^{-\beta \hat{k} \times \hat{k} \times} \right) \\
& = \frac{1}{2} \left(e^{(\alpha + \beta) \hat{k} \times \hat{k} \times} + e^{-(\alpha + \beta) \hat{k} \times \hat{k} \times} \right) \\
& = \text{coskk}(\alpha + \beta)
\end{aligned}$$

In diadically expanded matrix operators

$$\begin{aligned}
\mathbf{coskk}\alpha \mathbf{coskk}\beta + (\mathbf{sinkk}\alpha \hat{k} \times \hat{k} \times)(\mathbf{sinkk}\beta \hat{k} \times \hat{k} \times) &= \begin{pmatrix} \cosh\alpha & 0 & 0 \\ 0 & \cosh\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\beta & 0 & 0 \\ 0 & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\quad + \begin{pmatrix} -\sinh\alpha & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sinh\beta & 0 & 0 \\ 0 & -\sinh\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cosh\alpha \cosh\beta & 0 & 0 \\ 0 & \cosh\alpha \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \sinh\alpha \sinh\beta & 0 & 0 \\ 0 & \sinh\alpha \sinh\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \mathbf{coskk}(\alpha + \beta)
\end{aligned}$$

which is the same as the result for $\mathbf{coskk}$ from vector algebra and so all works as it should.

Changing base to the other principal rotation axes:

k -hat to j -hat principal rotation axis (xy plane to zx plane): changing base to the j -hat rotation axes with $\mathbf{expi}(\pi/2)$ and $\mathbf{expi}(-\pi/2)$

$$\begin{aligned}
\mathbf{coskk}(\alpha + \beta) &= \mathbf{coskk}\alpha \mathbf{coskk}\beta + (\mathbf{sinkk}\alpha \hat{k} \times \hat{k} \times)(\mathbf{sinkk}\beta \hat{k} \times \hat{k} \times) \quad \text{compound angle formula for } \mathbf{coskk} \\
\mathbf{expi}(\pi/2)(\mathbf{coskk}(\alpha + \beta))\mathbf{expi}(-\pi/2) &= \mathbf{expi}(\pi/2)(\mathbf{coskk}\alpha \mathbf{coskk}\beta + (\mathbf{sinkk}\alpha \hat{k} \times \hat{k} \times)(\mathbf{sinkk}\beta \hat{k} \times \hat{k} \times))\mathbf{expi}(-\pi/2) \quad \text{change of base} \\
\mathbf{cosjj}(\alpha + \beta) &= \mathbf{expi}(\pi/2)(\mathbf{coskk}\alpha \mathbf{coskk}\beta)\mathbf{expi}(-\pi/2) \\
&\quad + \mathbf{expi}(\pi/2)(\mathbf{sinkk}\alpha \hat{k} \times \hat{k} \times)(\mathbf{sinkk}\beta \hat{k} \times \hat{k} \times)\mathbf{expi}(-\pi/2) \\
&= \mathbf{expi}(\pi/2)(\mathbf{coskk}\alpha \mathbf{expi}(-\pi/2)\mathbf{expi}(\pi/2)\mathbf{coskk}\beta)\mathbf{expi}(-\pi/2) \\
&\quad + \mathbf{expi}(\pi/2)(\mathbf{sinkk}\alpha \hat{k} \times \hat{k} \times)\mathbf{expi}(-\pi/2)\mathbf{expi}(\pi/2)(\mathbf{sinkk}\beta \hat{k} \times \hat{k} \times)\mathbf{expi}(-\pi/2) \\
&= \mathbf{expi}(\pi/2)(\mathbf{coskk}\alpha \mathbf{expi}(-\pi/2)\mathbf{expi}(\pi/2)\mathbf{coskk}\beta)\mathbf{expi}(-\pi/2) \\
&\quad + \mathbf{expi}(\pi/2)(\mathbf{sinkk}\alpha \hat{k} \times \hat{k} \times)\mathbf{expi}(-\pi/2)\mathbf{expi}(\pi/2)(\mathbf{sinkk}\beta \hat{k} \times \hat{k} \times)\mathbf{expi}(-\pi/2) \\
&= \mathbf{cosjj}\alpha \mathbf{cosjj}\beta + (\mathbf{sinjj}\alpha \hat{j} \times \hat{j} \times)(\mathbf{sinjj}\beta \hat{j} \times \hat{j} \times) \quad \text{compound angle formula for } \mathbf{cosjj}
\end{aligned}$$

In diadically expanded matrix form,

$$\begin{aligned}
\mathbf{expi}(\pi/2)(\mathbf{coskk}(\alpha + \beta))\mathbf{expi}(-\pi/2) &= \mathbf{expi}(\pi/2)(\mathbf{coskk}\alpha \mathbf{coskk}\beta + (\mathbf{sinkk}\alpha \hat{k} \times \hat{k} \times)(\mathbf{sinkk}\beta \hat{k} \times \hat{k} \times))\mathbf{expi}(-\pi/2) \\
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cosh\alpha & 0 & 0 \\ 0 & \cosh\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\beta & 0 & 0 \\ 0 & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\sinh\alpha & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sinh\beta & 0 & 0 \\ 0 & -\sinh\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
\begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cosh\alpha & 0 & 0 \\ 0 & \cosh\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cosh\beta & 0 & 0 \\ 0 & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\sinh\alpha & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\sinh\beta & 0 & 0 \\ 0 & -\sinh\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \alpha \end{pmatrix} \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \beta \end{pmatrix} + \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \alpha \end{pmatrix} \begin{pmatrix} -\sinh \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \beta \end{pmatrix}$$

$$\mathbf{cosjj}(\alpha + \beta) = \mathbf{cosjj} \alpha \mathbf{cosjj} \beta + (\mathbf{sinjj} \alpha \hat{j} \times \hat{j} \times)(\mathbf{sinjj} \beta \hat{j} \times \hat{j} \times)$$

which is the required relation for the cosh compound argument formula for the j -hat rotation axis (for **cosjj**) base changed from the k -hat rotation axis (from **coskk**).

k -hat to i -hat principal rotation axis (xy plane to yz plane): changing base to the i -hat principal rotation axes with **expj** $(-\pi/2)$ and **expj** $(\pi/2)$

$$\begin{aligned} \mathbf{coskk}(\alpha + \beta) &= \mathbf{coskk} \alpha \mathbf{coskk} \beta + (\mathbf{sinkk} \alpha \hat{k} \times \hat{k} \times)(\mathbf{sinkk} \beta \hat{k} \times \hat{k} \times) && \text{compound angle formula for } \mathbf{coskk} \\ \mathbf{expj}(-\pi/2)(\mathbf{coskk}(\alpha + \beta))\mathbf{expj}(\pi/2) &= \mathbf{expj}(-\pi/2)(\mathbf{coskk} \alpha \mathbf{coskk} \beta + (\mathbf{sinkk} \alpha \hat{k} \times \hat{k} \times)(\mathbf{sinkk} \beta \hat{k} \times \hat{k} \times))\mathbf{expj}(\pi/2) && \text{change of base} \\ \mathbf{cosii}(\alpha + \beta) &= \mathbf{expj}(-\pi/2)(\mathbf{coskk} \alpha \mathbf{coskk} \beta)\mathbf{expj}(\pi/2) \\ &\quad + \mathbf{expj}(-\pi/2)(\mathbf{sinkk} \alpha \hat{k} \times \hat{k} \times)(\mathbf{sinkk} \beta \hat{k} \times \hat{k} \times)\mathbf{expj}(\pi/2) \\ &= \mathbf{expj}(-\pi/2)(\mathbf{coskk} \alpha \mathbf{expj}(\pi/2)\mathbf{expj}(-\pi/2)\mathbf{coskk} \beta)\mathbf{expj}(\pi/2) \\ &\quad + \mathbf{expj}(-\pi/2)(\mathbf{sinkk} \alpha \hat{k} \times \hat{k} \times)\mathbf{expj}(\pi/2)\mathbf{expj}(-\pi/2)(\mathbf{sinkk} \beta \hat{k} \times \hat{k} \times)\mathbf{expj}(\pi/2) \\ &= \mathbf{expj}(-\pi/2)(\mathbf{coskk} \alpha \mathbf{expj}(\pi/2)\mathbf{expj}(-\pi/2)\mathbf{coskk} \beta)\mathbf{expj}(\pi/2) \\ &\quad + \mathbf{expj}(-\pi/2)(\mathbf{sinkk} \alpha \hat{k} \times \hat{k} \times)\mathbf{expj}(\pi/2)\mathbf{expj}(-\pi/2)(\mathbf{sinkk} \beta \hat{k} \times \hat{k} \times)\mathbf{expj}(\pi/2) \\ &= \mathbf{cosii} \alpha \mathbf{cosii} \beta + (\mathbf{sinii} \alpha \hat{i} \times \hat{i} \times)(\mathbf{sinii} \beta \hat{i} \times \hat{i} \times) && \text{compound angle formula for } \mathbf{cosii} \end{aligned}$$

In diadically expanded matrix form,

$$\begin{aligned} \mathbf{expj}(-\pi/2)(\mathbf{coskk}(\alpha + \beta))\mathbf{expj}(\pi/2) &= \mathbf{expj}(-\pi/2)(\mathbf{coskk} \alpha \mathbf{coskk} \beta + (\mathbf{sinkk} \alpha \hat{k} \times \hat{k} \times)(\mathbf{sinkk} \beta \hat{k} \times \hat{k} \times))\mathbf{expj}(\pi/2) \\ \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & -\sinh \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sinh \beta & 0 & 0 \\ 0 & -\sinh \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & -\sinh \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sinh \beta & 0 & 0 \\ 0 & -\sinh \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & \cosh \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & \cosh \beta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh \alpha & 0 \\ 0 & 0 & -\sinh \alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh \beta & 0 \\ 0 & 0 & -\sinh \beta \end{pmatrix} \\ \mathbf{cosii}(\alpha + \beta) &= \mathbf{cosii} \alpha \mathbf{cosii} \beta + (\mathbf{sinii} \alpha \hat{i} \times \hat{i} \times)(\mathbf{sinii} \beta \hat{i} \times \hat{i} \times) \end{aligned}$$

which is the required relation for the cosh compound argument formula for the i -hat rotation axis (for **cosii**) base changed from the k -hat rotation axis (from **coskk**).

The relations for yz plane ($\hat{j} \times$ rotation axis) basis sines, cosines, and tangents (**sinjj**, **cosjj**, and **tanj**) are

- (A) Hyperbolic Euler form for **cisjj** = **expjj** **cosjj** θ - **sinjj** $\theta \hat{j} \times \hat{j} \times$ = **expjj** θ
 (B) Hyperbolic conjugate Euler form **cosjj** θ + **sinjj** $\theta \hat{j} \times \hat{j} \times$ = **expjj** $(-\theta)$
 (1) Basic hyperbolic relation for **sinjj** and **cosjj** **cosjj** θ^2 - (**sinjj** $\theta \hat{j} \times \hat{j} \times$) 2 = 1

(2) Basic hyperbolic relation for tanjj and secjj	$1 - (\tan jj \theta \hat{j} \times \hat{j} \times)^2 = \sec jj^2 \theta$
(3) Compound angle formula for sinjj	$\sin jj(\alpha + \beta) \hat{j} \times \hat{j} \times = \cos jj \alpha \sin jj \beta \hat{j} \times \hat{j} \times + \cos jj \beta \sin jj \alpha \hat{j} \times \hat{j} \times$
(4) Compound angle formula for cosjj	$\cos jj(\alpha + \beta) = \cos jj \alpha \cos jj \beta + (\sin jj \alpha \hat{j} \times \hat{j} \times)(\sin jj \beta \hat{j} \times \hat{j} \times)$
(5) Compound angle formula for tanjj	$\tan j(\alpha + \beta) \hat{j} \times \hat{j} \times = \frac{\tan j \alpha \hat{j} \times \hat{j} \times + \tan j \beta \hat{j} \times \hat{j} \times}{1 + (\tan j \alpha \hat{j} \times \hat{j} \times)(\tan j \beta \hat{j} \times \hat{j} \times)}$
(6) Double angle formula for sinjj	$\sin jj(2\alpha) \hat{j} \times \hat{j} \times = 2 \cos jj \alpha \sin jj \alpha \hat{j} \times \hat{j} \times$
(7) Double angle formula for cosjj	$\cos jj(2\alpha) = \cos jj^2 \alpha + (\sin jj \alpha \hat{j} \times \hat{j} \times)^2$
(8) Double angle formula for tanjj	$\tan jj(2\alpha) \hat{j} \times \hat{j} \times = (2 \tan jj \alpha \hat{j} \times \hat{j} \times) (1 + (\tan jj \alpha \hat{j} \times \hat{j} \times)^2)^{-1}$
(9) Products of sinjj	$2(\sin jj \alpha \hat{j} \times \hat{j} \times)(\sin jj \beta \hat{j} \times \hat{j} \times) = \cos jj(\alpha + \beta) - \cos jj(\alpha - \beta)$
(10) Products of cosjj	$2 \cos jj \alpha \cos jj \beta = \cos jj(\alpha + \beta) + \cos jj(\alpha - \beta)$
(11) Products of sinjj and cosjj	$2(\sin jj \alpha \hat{j} \times \hat{j} \times) \cos jj \beta = \sin jj(\alpha + \beta) \hat{j} \times \hat{j} \times + \sin jj(\alpha - \beta) \hat{j} \times \hat{j} \times$

Diadically expanded into matrix base for the $\hat{j}$ -hat principal rotation axis:

nb, **sinjj** is coupled with the double cross $\hat{j} \times \hat{j} \times$ (similarly as coupling **sinj** with $\hat{j} \times$)

$$\sin jj \theta = \begin{pmatrix} \sinh \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sinh \theta \end{pmatrix} \quad \sin jj \theta \hat{j} \times \hat{j} \times = \begin{pmatrix} \sinh \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sinh \theta \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -\sinh \theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \theta \end{pmatrix}$$

(A) Hyperbolic Euler form for **cisjj = **expjj****

$$\cosh \theta + \sinh \theta = e^\theta \quad \cos jj \theta - \sin jj \theta \hat{j} \times \hat{j} \times = \exp jj \theta$$

$$\begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \theta \end{pmatrix} - \begin{pmatrix} -\sinh \theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \theta \end{pmatrix} = \begin{pmatrix} e^\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^\theta \end{pmatrix}$$

(B) Hyperbolic conjugate Euler form

$$\cosh \theta - \sinh \theta = e^{-\theta} \quad \cos jj \theta + \sin jj \theta \hat{j} \times \hat{j} \times = \exp jj(-\theta)$$

$$\begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \theta \end{pmatrix} + \begin{pmatrix} -\sinh \theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \theta \end{pmatrix} = \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\theta} \end{pmatrix}$$

(1) Basic circular relation for **sinjj and **cosjj****

$$\cosh^2 \theta - \sinh^2 \theta = 1 \quad \cos jj^2 \theta - (\sin jj \theta \hat{j} \times \hat{j} \times)^2 = 1$$

$$\begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \theta \end{pmatrix}^2 - \begin{pmatrix} -\sinh \theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \theta \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) Basic circular relation for **tanjj** and **secjj**

$$1 - \tanh^2 \theta = \operatorname{sech}^2 \theta$$

$$1 - (\mathbf{tanjj} \theta \hat{j} \times \hat{j} \times)^2 = \mathbf{secjj}^2 \theta$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -\tanh \theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tanh \theta \end{pmatrix}^2 = \begin{pmatrix} \operatorname{sech} \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \operatorname{sech} \theta \end{pmatrix}^2$$

(3) Compound angle formula for **sinjj**

$$\sinh(\alpha + \beta) = \sinh \alpha \cosh \beta + \cosh \alpha \sinh \beta$$

$$\mathbf{sinjj}(\alpha + \beta) \hat{j} \times = \mathbf{cosjj} \alpha \mathbf{sinjj} \beta \hat{j} \times \hat{j} \times + \mathbf{cosjj} \beta \mathbf{sinjj} \alpha \hat{j} \times \hat{j} \times$$

$$\begin{pmatrix} -\sinh(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \alpha \end{pmatrix} \begin{pmatrix} -\sinh \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \beta \end{pmatrix} + \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \beta \end{pmatrix} \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \alpha \end{pmatrix}$$

(4) Compound angle formula for **cosjj**

$$\cosh(\alpha + \beta) = \cosh \alpha \cosh \beta + \sinh \alpha \sinh \beta$$

$$\mathbf{cosjj}(\alpha + \beta) = \mathbf{cosjj} \alpha \mathbf{cosjj} \beta + (\mathbf{sinjj} \alpha \hat{j} \times \hat{j} \times)(\mathbf{sinjj} \beta \hat{j} \times \hat{j} \times)$$

$$\begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \alpha \end{pmatrix} \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \beta \end{pmatrix} + \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \alpha \end{pmatrix} \begin{pmatrix} -\sinh \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \beta \end{pmatrix}$$

(5) Compound angle formula for **tanjj**

$$\tanh(\alpha + \beta) = (\tanh \alpha + \tanh \beta)(1 + \tanh \alpha \tanh \beta)^{-1} \quad \mathbf{tanjj}(\alpha + \beta) \hat{j} \times \hat{j} \times = \frac{\mathbf{tanjj} \alpha \hat{j} \times \hat{j} \times + \mathbf{tanjj} \beta \hat{j} \times \hat{j} \times}{1 + (\mathbf{tanjj} \alpha \hat{j} \times \hat{j} \times)(\mathbf{tanjj} \beta \hat{j} \times \hat{j} \times)}$$

$$\begin{pmatrix} -\tanh(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tanh(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} -\tanh \alpha - \tanh \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tanh \alpha - \tanh \beta \end{pmatrix} \begin{pmatrix} 1 + \tanh \alpha \tanh \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + \tanh \alpha \tanh \beta \end{pmatrix}^{-1}$$

(6) Double angle formula for **sinjj**

$$\sinh(2\alpha) = 2 \sinh \alpha \cosh \alpha$$

$$\mathbf{sinjj}(2\alpha) \hat{j} \times \hat{j} \times = 2 \mathbf{cosjj} \alpha \mathbf{sinjj} \alpha \hat{j} \times \hat{j} \times$$

$$\begin{pmatrix} -\sinh 2\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh 2\alpha \end{pmatrix} = 2 \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \alpha \end{pmatrix} \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \alpha \end{pmatrix}$$

(7) Double angle formula for **cosjj**

$$\cosh(2\alpha) = \cosh^2 \alpha + \sinh^2 \alpha$$

$$\mathbf{cosjj}(2\alpha) = \mathbf{cosjj}^2 \alpha + (\mathbf{sinjj} \alpha \hat{j} \times \hat{j} \times)^2$$

$$\begin{pmatrix} \cosh 2\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh 2\alpha \end{pmatrix} = \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \alpha \end{pmatrix}^2 + \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \alpha \end{pmatrix}^2$$

(8) Double angle formula for **tanjj**

$$\tanh(2\alpha) = 2\tanh\alpha(1 + \tanh^2\alpha)^{-1} \quad \mathbf{tanjj}(2\alpha) \hat{j} \times \hat{j} \times = (2\mathbf{tanjj}\alpha \hat{j} \times \hat{j} \times) (1 + (\mathbf{tanjj}\alpha \hat{j} \times \hat{j} \times)^2)^{-1}$$

$$\begin{pmatrix} -\tanh 2\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tanh 2\alpha \end{pmatrix} = 2 \begin{pmatrix} -\tanh \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tanh \alpha \end{pmatrix} \begin{pmatrix} 1 + \tanh^2 \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + \tanh^2 \alpha \end{pmatrix}^{-1}$$

(9) Products of **sinjj** (inverse formulae from addition formulae)

$$2\sinh\alpha\sinh\beta = \cosh(\alpha + \beta) - \cosh(\alpha - \beta) \quad 2(\mathbf{sinjj}\alpha \hat{j} \times \hat{j} \times)(\mathbf{sinjj}\beta \hat{j} \times \hat{j} \times) = \mathbf{cosjj}(\alpha + \beta) - \mathbf{cosjj}(\alpha - \beta)$$

$$2 \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \alpha \end{pmatrix} \begin{pmatrix} -\sinh \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \beta \end{pmatrix} = \begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} - \begin{pmatrix} \cosh(\alpha - \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh(\alpha - \beta) \end{pmatrix}$$

(10) Products of **cosjj**

$$2\cosh\alpha\cosh\beta = \cosh(\alpha + \beta) + \cosh(\alpha - \beta) \quad 2\mathbf{cosjj}\alpha \mathbf{cosjj}\beta = \mathbf{cosjj}(\alpha + \beta) + \mathbf{cosjj}(\alpha - \beta)$$

$$2 \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \alpha \end{pmatrix} \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \beta \end{pmatrix} = \begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} + \begin{pmatrix} \cosh(\alpha - \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh(\alpha - \beta) \end{pmatrix}$$

(11) Products of **sinjj** and **cosjj**

$$2\sinh\alpha\cosh\beta = \sinh(\alpha + \beta) + \sinh(\alpha - \beta) \quad 2(\mathbf{sinjj}\alpha \hat{j} \times \hat{j} \times) \mathbf{cosjj}\beta = \mathbf{sinjj}(\alpha + \beta) \hat{j} \times \hat{j} \times + \mathbf{sinjj}(\alpha - \beta) \hat{j} \times \hat{j} \times$$

$$2 \begin{pmatrix} -\sinh \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh \alpha \end{pmatrix} \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \beta \end{pmatrix} = \begin{pmatrix} -\sinh(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh(\alpha + \beta) \end{pmatrix} + \begin{pmatrix} -\sinh(\alpha - \beta) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh(\alpha - \beta) \end{pmatrix}$$

The relations for zx plane ($\hat{i} \times$ rotation axis) basis sines, cosines, and tangents (**sinii**, **cosii**, and **tanii**) are

(A) Hyperbolic Euler form (for **cisii** = **expii**)

$$\mathbf{cosii}\theta - \mathbf{sinii}\theta \hat{i} \times \hat{i} \times = \mathbf{expii}\theta$$

(B) Hyperbolic conjugate Euler form

$$\mathbf{cosii}\theta + \mathbf{sinii}\theta \hat{i} \times \hat{i} \times = \mathbf{expii}(-\theta)$$

(1) Basic hyperbolic relation (for **sinii** and **cosii**)

$$\mathbf{cosii}^2\theta - (\mathbf{sinii}\theta \hat{i} \times \hat{i} \times)^2 = 1$$

(2) Basic hyperbolic relation (for **tanii** and **secii**)

$$1 - (\mathbf{tanii}\theta \hat{i} \times \hat{i} \times)^2 = \mathbf{secii}^2\theta$$

(3) Compound angle formula for **sinii**

$$\mathbf{sinii}(\alpha + \beta) \hat{i} \times \hat{i} \times = \mathbf{cosii}\alpha \mathbf{sinii}\beta \hat{i} \times \hat{i} \times + \mathbf{cosii}\beta \mathbf{sinii}\alpha \hat{i} \times \hat{i} \times$$

(4) Compound angle formula for **cosii**

$$\mathbf{cosii}(\alpha + \beta) = \mathbf{cosii}\alpha \mathbf{cosii}\beta + (\mathbf{sinii}\alpha \hat{i} \times \hat{i} \times)(\mathbf{sinii}\beta \hat{i} \times \hat{i} \times)$$

(5) Compound angle formula for **tanii**

$$\mathbf{tanii}(\alpha + \beta) \hat{i} \times \hat{i} \times = \frac{\mathbf{tanii}\alpha \hat{i} \times \hat{i} \times + \mathbf{tanii}\beta \hat{i} \times \hat{i} \times}{1 + (\mathbf{tanii}\alpha \hat{i} \times \hat{i} \times)(\mathbf{tanii}\beta \hat{i} \times \hat{i} \times)}$$

(6) Double angle formula for **sinii**

$$\mathbf{sinii}(2\alpha) \hat{i} \times \hat{i} \times = 2\mathbf{cosii}\alpha \mathbf{sinii}\alpha \hat{i} \times \hat{i} \times$$

(7) Double angle formula for **cosii**

$$\mathbf{cosii}(2\alpha) = \mathbf{cosii}^2\alpha + (\mathbf{sinii}\alpha \hat{i} \times \hat{i} \times)^2$$

(8) Double angle formula for **tanii**

$$\mathbf{tanii}(2\alpha) \hat{i} \times \hat{i} \times = (2\mathbf{tanii}\alpha \hat{i} \times \hat{i} \times) (1 + (\mathbf{tanii}\alpha \hat{i} \times \hat{i} \times)^2)^{-1}$$

(9) Products of **sinii**

$$2(\mathbf{sinii}\alpha \hat{i} \times \hat{i} \times)(\mathbf{sinii}\beta \hat{i} \times \hat{i} \times) = \mathbf{cosii}(\alpha + \beta) - \mathbf{cosii}(\alpha - \beta)$$

(10) Products of **cosii**

$$2\mathbf{cosii}\alpha \mathbf{cosii}\beta = \mathbf{cosii}(\alpha + \beta) + \mathbf{cosii}(\alpha - \beta)$$

(11) Products of **sinii** and **cosii**

$$2(\mathbf{sinii}\alpha \hat{i} \times \hat{i} \times) \mathbf{cosii}\beta = \mathbf{sinii}(\alpha + \beta) \hat{i} \times \hat{i} \times + \mathbf{sinii}(\alpha - \beta) \hat{i} \times \hat{i} \times$$

Diadically expanded into matrix base for the $\hat{i}$ -hat principal rotation axis:

nb, **sinii** is coupled with the double cross $\hat{i} \times \hat{i} \times$ (similarly as coupling **sinii** with $\hat{i} \times$)

$$\mathbf{sinii}\theta = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sinh\theta & 0 \\ 0 & 0 & \sinh\theta \end{pmatrix} \quad \mathbf{sinii}\theta \hat{i} \times \hat{i} \times = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sinh\theta & 0 \\ 0 & 0 & \sinh\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & -\sinh\theta \end{pmatrix}$$

(A) Hyperbolic Euler form for **cisii = **expii****

$$\cosh\theta + \sinh\theta = e^\theta \quad \mathbf{cosii}\theta - \mathbf{sinii}\theta \hat{i} \times \hat{i} \times = \mathbf{expii}\theta$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & -\sinh\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^\theta & 0 \\ 0 & 0 & e^\theta \end{pmatrix}$$

(B) Hyperbolic conjugate Euler form

$$\cosh\theta - \sinh\theta = e^{-\theta} \quad \mathbf{cosii}\theta + \mathbf{sinii}\theta \hat{i} \times \hat{i} \times = \mathbf{expii}(-\theta)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & -\sinh\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & e^{-\theta} \end{pmatrix}$$

(1) Basic circular relation for **sinii and **cosii****

$$\cosh^2\theta - \sinh^2\theta = 1 \quad \mathbf{cosii}^2\theta - (\mathbf{sinii}\theta \hat{i} \times \hat{i} \times)^2 = 1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix}^2 - \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & -\sinh\theta \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) Basic circular relation for **tanii and **secii****

$$1 - \tanh^2\theta = \operatorname{sech}^2\theta \quad 1 - (\mathbf{tanii}\theta \hat{i} \times \hat{i} \times)^2 = \mathbf{secii}^2\theta$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\tanh\theta & 0 \\ 0 & 0 & -\tanh\theta \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \operatorname{sech}\theta & 0 \\ 0 & 0 & \operatorname{sech}\theta \end{pmatrix}^2$$

(3) Compound angle formula for **sinii**

$$\sinh(\alpha + \beta) = \sinh\alpha \cosh\beta + \cosh\alpha \sinh\beta \quad \mathbf{sinii}(\alpha + \beta) \hat{i} \times \hat{i} \times = \mathbf{cosii}\alpha \mathbf{sinii}\beta \hat{i} \times \hat{i} \times + \mathbf{cosii}\beta \mathbf{sinii}\alpha \hat{i} \times \hat{i} \times$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh(\alpha + \beta) & 0 \\ 0 & 0 & -\sinh(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\alpha & 0 \\ 0 & 0 & \cosh\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\beta & 0 \\ 0 & 0 & -\sinh\beta \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\beta & 0 \\ 0 & 0 & \cosh\beta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & -\sinh\alpha \end{pmatrix}$$

(4) Compound angle formula for **cosii**

$$\cosh(\alpha + \beta) = \cosh\alpha \cosh\beta + \sinh\alpha \sinh\beta$$

$$\mathbf{cosii}(\alpha + \beta) = \mathbf{cosii}\alpha \mathbf{cosii}\beta + (\mathbf{sinii}\alpha \hat{i} \times \hat{i} \times)(\mathbf{sinii}\beta \hat{i} \times \hat{i} \times)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\alpha & 0 \\ 0 & 0 & \cosh\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\beta & 0 \\ 0 & 0 & \cosh\beta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & -\sinh\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\beta & 0 \\ 0 & 0 & -\sinh\beta \end{pmatrix}$$

(5) Compound angle formula for **tanii**

$$\tanh(\alpha + \beta) = (\tanh\alpha + \tanh\beta)(1 + \tanh\alpha \tanh\beta)^{-1} \quad \mathbf{tanii}(\alpha + \beta) \hat{i} \times \hat{i} \times = \frac{\mathbf{tanii}\alpha \hat{i} \times \hat{i} \times + \mathbf{tanii}\beta \hat{i} \times \hat{i} \times}{1 + (\mathbf{tanii}\alpha \hat{i} \times \hat{i} \times)(\mathbf{tanii}\beta \hat{i} \times \hat{i} \times)}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\tanh(\alpha + \beta) & 0 \\ 0 & 0 & -\tanh(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\tanh\alpha - \tanh\beta & 0 \\ 0 & 0 & -\tanh\alpha - \tanh\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + \tanh\alpha \tanh\beta & 0 \\ 0 & 0 & 1 + \tanh\alpha \tanh\beta \end{pmatrix}^{-1}$$

(6) Double angle formula for **sinii**

$$\sinh(2\alpha) = 2\sinh\alpha \cosh\alpha$$

$$\mathbf{sinii}(2\alpha) \hat{i} \times \hat{i} \times = 2\mathbf{cosii}\alpha \mathbf{sinii}\alpha \hat{i} \times \hat{i} \times$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh 2\alpha & 0 \\ 0 & 0 & -\sinh 2\alpha \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\alpha & 0 \\ 0 & 0 & \cosh\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & -\sinh\alpha \end{pmatrix}$$

(7) Double angle formula for **cosii**

$$\cosh 2\alpha = \cosh^2 \alpha + \sinh^2 \alpha$$

$$\mathbf{cosii}(2\alpha) = \mathbf{cosii}^2 \alpha + (\mathbf{sinii}\alpha \hat{i} \times \hat{i} \times)^2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh 2\alpha & 0 \\ 0 & 0 & \cosh 2\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\alpha & 0 \\ 0 & 0 & \cosh\alpha \end{pmatrix}^2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & -\sinh\alpha \end{pmatrix}^2$$

(8) Double angle formula for **tanii**

$$\tanh(2\alpha) = 2\tanh\alpha(1 + \tanh^2 \alpha)^{-1}$$

$$\mathbf{tanii}(2\alpha) \hat{i} \times \hat{i} \times = (2\mathbf{tanii}\alpha \hat{i} \times \hat{i} \times)(1 + (\mathbf{tanii}\alpha \hat{i} \times \hat{i} \times)^2)^{-1}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\tanh 2\alpha & 0 \\ 0 & 0 & -\tanh 2\alpha \end{pmatrix} = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\tanh\alpha & 0 \\ 0 & 0 & -\tanh\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + \tanh^2 \alpha & 0 \\ 0 & 0 & 1 + \tanh^2 \alpha \end{pmatrix}^{-1}$$

(9) Products of **sinii** (inverse formulae from addition formulae)

$$2\sinh\alpha \sinh\beta = \cosh(\alpha + \beta) - \cosh(\alpha - \beta)$$

$$2(\mathbf{sinii}\alpha \hat{i} \times \hat{i} \times)(\mathbf{sinii}\beta \hat{i} \times \hat{i} \times) = \mathbf{cosii}(\alpha + \beta) - \mathbf{cosii}(\alpha - \beta)$$

$$2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & -\sinh\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\beta & 0 \\ 0 & 0 & -\sinh\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\alpha - \beta) & 0 \\ 0 & 0 & \cosh(\alpha - \beta) \end{pmatrix}$$

(10) Products of **cosii**

$$2\cosh\alpha\cosh\beta = \cosh(\alpha + \beta) + \cosh(\alpha - \beta)$$

$$2\mathbf{cosii}\alpha\mathbf{cosii}\beta = \mathbf{cosii}(\alpha + \beta) + \mathbf{cosii}(\alpha - \beta)$$

$$2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\alpha & 0 \\ 0 & 0 & \cosh\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\beta & 0 \\ 0 & 0 & \cosh\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\alpha - \beta) & 0 \\ 0 & 0 & \cosh(\alpha - \beta) \end{pmatrix}$$

(11) Products of **sinii** and **cosii**

$$2\sinh\alpha\cosh\beta = \sinh(\alpha + \beta) + \sinh(\alpha - \beta)$$

$$2(\mathbf{sinii}\alpha\hat{i} \times \hat{i} \times)\mathbf{cosii}\beta = \mathbf{sinii}(\alpha + \beta)\hat{i} \times \hat{i} \times + \mathbf{sinii}(\alpha - \beta)\hat{i} \times \hat{i} \times$$

$$2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh\alpha & 0 \\ 0 & 0 & -\sinh\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\beta & 0 \\ 0 & 0 & \cosh\beta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh(\alpha + \beta) & 0 \\ 0 & 0 & -\sinh(\alpha + \beta) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sinh(\alpha - \beta) & 0 \\ 0 & 0 & -\sinh(\alpha - \beta) \end{pmatrix}$$

Like all the formulae in functions of $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$, these also hold for the polar cylindrical and polar spherical bases such as $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ and their quaternion (complex number) operators $\hat{r} \times$, $\hat{\theta} \times$, and $\hat{\phi} \times$ (including the double crosses $\hat{r} \times \hat{r} \times$, $\hat{\theta} \times \hat{\theta} \times$ and $\hat{\phi} \times \hat{\phi} \times$ for centripetal forces in polars for example or for radar return echoes oriented over azimuth and elevation).

Identities of $\sinh k$ and $\cosh k$: Hyperbolic Relations of $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$

Axis Osculating Hyperbolic Sine and Cosine – Relations of $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$, $\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times$, and $\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times$.

Note that all of the trigonometric expressions with $i = \hat{k} \times$ can alternatively be written with $\hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times$ because of the cross product identity $\hat{k} \times = \hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times$, i.e., because the circular complex number (rotation operator) for the k -hat rotation axis can be expressed alternatively as $i = \hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times$ from the identity $\hat{k} \times = \hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times$. If the negative sign in $\hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times$ is changed to a plus, $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$, then the circular complex number is converted into the hyperbolic complex number. Unfortunately, this cross product operator doesn't have a simpler form as of this time and so is large and bulky. This makes the basis operator large and unwieldy. It still works out perfectly but the equations are large. The diadically expanded operators, though, are the same size as those for $\hat{k} \times$ and $\hat{k} \times \hat{k} \times$ and so do not pose a problem in a diadically expanded matrix basis. With this caveat in mind, the algebraic size of $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$, the identities for the axis osculating hyperbolics are now derived and tabulated. In contrast to the diagonal attenuation matrices of the double cross hyperbolics, these are the hyperbolic equivalents to the rotation matrices. For the double cross basis exponentials **expkk**, **expjj**, and **expii** the hyperbolic signs remain on the main diagonal). So the principal difference here, with the axis osculating rather than sign osculating hyperbolics, is that the scalar sinh functions for **expkh**, **expjh**, and **expih** are located on the non-diagonals similarly as the scalar sines in **expk**, **expj**, and **expi**.

Basis Hyperbolic Sine and Cosine (Quaternion Hyperbolic Sine and Cosine) Identities

Trigonometric Identities of $\sinh k$ and $\cosh k$

$$\begin{aligned} \cosh k \theta &= \frac{1}{2}(\mathbf{expkh}(\theta) + \mathbf{expkh}(-\theta)) \\ &= \frac{1}{2}(e^{\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)} + e^{-\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)}) \end{aligned}$$

$$\begin{aligned} \sinh k \theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) &= \frac{1}{2}(\mathbf{expkh}(\theta) - \mathbf{expkh}(-\theta)) \\ &= \frac{1}{2}(e^{\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)} - e^{-\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)}) \end{aligned}$$

$$\begin{aligned} \cosh j \theta &= \frac{1}{2}(\mathbf{expjh}(\theta) + \mathbf{expjh}(-\theta)) \\ &= \frac{1}{2}(e^{\theta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)} + e^{-\theta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)}) \end{aligned}$$

$$\begin{aligned} \sinh j \theta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) &= \frac{1}{2}(\mathbf{expjh}(\theta) - \mathbf{expjh}(-\theta)) \\ &= \frac{1}{2}(e^{\theta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)} - e^{-\theta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)}) \end{aligned}$$

and

$$\begin{aligned} \cosh \theta &= \frac{1}{2}(\text{exphi}(\theta) + \text{exphi}(-\theta)) & \sinh \theta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) &= \frac{1}{2}(\text{expi}(\theta) - \text{expi}(-\theta)) \\ &= \frac{1}{2}(e^{\theta(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)} + e^{-\theta(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)}) & &= \frac{1}{2}(e^{\theta(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)} - e^{-\theta(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)}) \end{aligned}$$

Negative arguments of coshk and sinhk (and tanhk)

Corresponding to $\cos(-\theta) = \cos \theta$, $\sin(-\theta) = -\sin \theta$, and $\tan(-\theta) = -\tan \theta$

$$\begin{aligned} \cosh k(-\theta) &= \cosh k \theta \\ \sinh k(-\theta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) &= -\sinh k \theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \\ \tanh k(-\theta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) &= -\tanh k \theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \end{aligned}$$

for the k -hat xy orthogonal plane, as well as for j -hat and i -hat (zx and yz planes)

$$\begin{aligned} \cosh j(-\theta) &= \cosh j \theta & \cosh i(-\theta) &= \cosh i \theta \\ \sinh j(-\theta)(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) &= -\sinh j \theta (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) & \sinh i(-\theta)(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) &= -\sinh i \theta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) \\ \tanh j(-\theta)(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) &= -\tanh j \theta (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) & \tanh i(-\theta)(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) &= -\tanh i \theta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) \end{aligned}$$

For the xy -plane and the k -hat hyperbolic foci

$$\begin{aligned} \cosh k(-\theta) &= \frac{1}{2}(e^{(-\theta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)} + e^{-(-\theta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)}) \\ &= \frac{1}{2}(e^{-\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)} + e^{\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)}) = \frac{1}{2}(e^{\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)} + e^{-\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)}) \\ &= \cosh k \theta \end{aligned}$$

$$\cosh k(-\theta) = \begin{pmatrix} \cosh(-\theta) & 0 & 0 \\ 0 & \cosh(-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \cosh k \theta$$

$$\begin{aligned} \sinh k(-\theta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) &= \frac{1}{2}(e^{(-\theta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)} - e^{-(-\theta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)}) \\ &= \frac{1}{2}(e^{-\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)} - e^{\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)}) = -\frac{1}{2}(e^{\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)} - e^{-\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)}) \\ &= -\sinh k \theta \hat{k} \times \end{aligned}$$

$$\begin{aligned} \sinh k(-\theta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) &= \begin{pmatrix} 0 & -\sinh(-\theta) & 0 \\ \sinh(-\theta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sinh \theta & 0 \\ -\sinh \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -\sinh \theta & 0 \\ \sinh \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= -\sinh k \theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \end{aligned}$$

$$\tanh k(-\theta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = (\cosh k(-\theta))^{-1} \sinh k(-\theta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = (\cosh k \theta)^{-1} \sinh k \theta \hat{k} \times = \tanh k \theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)$$

$$\begin{aligned} \tanh k(-\theta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) &= \begin{pmatrix} 0 & -\tanh(-\theta) & 0 \\ \tanh(-\theta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tanh \theta & 0 \\ -\tanh \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -\tanh \theta & 0 \\ \tanh \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= -\tanh k \theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \end{aligned}$$

Unlike for the hyperbolic double crosses, the periodic properties corresponding to $\cosh(\theta + 2\pi ni) = \cosh \theta$, $\sinh(\theta + 2\pi ni) = \sinh \theta$, and $\tanh(\theta + 2\pi ni) = \tanh \theta$ for the axis osculating hyperbolics are unclear at the present time. This is because left to right multiplication by $i = k \times$ of $i \times j \times + j \times i \times$ differs from that of the left to right multiplication by $i \times j \times + j \times i \times$ of $i = k \times$ within the hyperbolic argument

$$\hat{k} \times (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{whilst} \quad (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \hat{k} \times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For this reason, the periodic properties of the axis osculating hyperbolics are omitted [due to not at this time wanting to determine all of the details of the ordering of the operators in the hyperbolic argument].

Being similarly stated for the double cross hyperbolics, the scalar hyperbolic identities are

(A) Hyperbolic Euler form	$\cosh \theta + \sinh \theta = e^\theta$
(B) Hyperbolic Conjugate Euler form	$\cosh \theta - \sinh \theta = e^{-\theta}$
(1) Basic hyperbolic relation for sinh and cosh	$\cosh^2 \theta - \sinh^2 \theta = 1$
(2) Basic hyperbolic relation for tanh and sech	$1 - \tanh^2 \theta = \text{sech}^2 \theta$
(3) Compound angle formula for sinh	$\sinh(\alpha + \beta) = \sinh \alpha \cosh \beta + \cosh \alpha \sinh \beta$
(4) Compound angle formula for cosh	$\cosh(\alpha + \beta) = \cosh \alpha \cosh \beta + \sinh \alpha \sinh \beta$
(5) Compound angle formula for tanh	$\tanh(\alpha + \beta) = (\tanh \alpha + \tanh \beta) (1 + \tanh \alpha \tanh \beta)^{-1}$
(6) Double angle formula for sinh	$\sinh(2\alpha) = 2 \sinh \alpha \cosh \alpha$
(7) Double angle formula for cosh	$\cosh(2\alpha) = \cosh^2 \alpha + \sinh^2 \alpha$
(8) Double angle formula for tanh	$\tanh(2\alpha) = 2 \tanh \alpha (1 + \tanh^2 \alpha)^{-1}$
(9) Products of sinh	$2 \sinh \alpha \sinh \beta = \cosh(\alpha + \beta) - \cosh(\alpha - \beta)$
(10) Products of cosh	$2 \cosh \alpha \cosh \beta = \cosh(\alpha + \beta) + \cosh(\alpha - \beta)$
(11) Products of sinh and cosh	$2 \sinh \alpha \cosh \beta = \sinh(\alpha + \beta) + \sinh(\alpha - \beta)$

The axis osculating hyperbolic complex numbers for each of the three R^3 orthogonal planes and hyperbolic foci are

$$\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for the xy plane}$$

$$\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{for the zx plane}$$

$$\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{for the yz plane}$$

These are the same diadically expanded operators of the circular complex numbers ($\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$) with the single difference of the absence of the sign change in one of the two curl positions. This is what makes these operators hyperbolic rather than circular. These operators then produce the axis osculating basis hyperbolics.

The axis osculating hyperbolic complex number for the plane orthogonal to the $\hat{k}$ -hat rotation axis is $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$. The relations for xy plane ($\hat{k} \times$ rotation axis) hyperbolic basis sines, cosines, and tangents (**sinhk**, **coshk**, and **tanhk**) are

(A) Hyperbolic Euler form	$\cosh k \theta + \sinh k \theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \mathbf{exp} k \theta$
(B) Hyperbolic Conjugate	$\cosh k \theta - \sinh k \theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \mathbf{exp} k (-\theta)$
(1) Basic hyperbolic relation	$\cosh^2 \theta - (\sinh k \theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times))^2 = 1$

- (2) Basic hyperbolic relation $1 - (\tanh k \theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times))^2 = \text{sech}^2 k \theta$
- (3) Compound angle formula $\sinh k(\alpha + \beta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \cosh k \alpha \sinh k \beta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) + \cosh k \beta \sinh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)$
- (4) Compound angle formula $\cosh k(\alpha + \beta) = \cosh k \alpha \cosh k \beta + \sinh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \sinh k \beta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)$
- (5) Compound angle formula $\tanh k(\alpha + \beta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \frac{\tanh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) + \tanh k \beta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)}{1 + \tanh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \tanh k \beta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)}$
- (6) Double angle formula $\sinh k(2\alpha)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = 2 \cosh k \alpha \sinh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)$
- (7) Double angle formula $\cosh k(2\alpha) = \cosh^2 k \alpha + (\sinh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times))^2$
- (8) Double angle formula $\tanh k(2\alpha)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = (2 \tanh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)) (1 + (\tanh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times))^2)^{-1}$
- (9) Products of $\sinh k$ $2 \sinh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \sinh k \beta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \cosh k(\alpha + \beta) - \cosh k(\alpha - \beta)$
- (10) Products of $\cosh k$ $2 \cosh k \alpha \cosh k \beta = \cosh k(\alpha + \beta) + \cosh k(\alpha - \beta)$
- (11) Products $\sinh k$ and $\cosh k$ $2 \sinh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \cosh k \beta = \sinh k(\alpha + \beta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) + \sinh k(\alpha - \beta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)$

Diadically expanded into matrix bases:

Noting here for reference that I have coupled $\sinh k$ with $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$

$$\sinh k \theta = \begin{pmatrix} \sinh \theta & 0 & 0 \\ 0 & \sinh \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \quad \sinh k \theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \begin{pmatrix} \sinh \theta & 0 & 0 \\ 0 & \sinh \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sinh \theta & 0 \\ \sinh \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(A) Hyperbolic Euler form for $\cosh k = \text{exphk}$

$$\cosh k \theta + \sinh k \theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \text{exphk} \theta$$

$$\begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \sinh \theta & 0 \\ \sinh \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh \theta & \sinh \theta & 0 \\ \sinh \theta & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(B) Hyperbolic Conjugate Euler form

$$\cosh k \theta - \sinh k \theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \text{exphk}(-\theta)$$

$$\begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \sinh \theta & 0 \\ \sinh \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh \theta & -\sinh \theta & 0 \\ -\sinh \theta & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(1) Basic hyperbolic relation for $\sinh k$ and $\cosh k$

$$\cosh^2 k \theta - (\sinh k \theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times))^2 = 1$$

$$\begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} 0 & \sinh \theta & 0 \\ \sinh \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) Basic hyperbolic relation for $\tanh k$ and $\operatorname{sech} k$

$$1 - \left(\tanh k \theta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \right)^2 = \operatorname{sech}^2 \theta$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \tanh \theta & 0 \\ \tanh \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} \operatorname{sech} \theta & 0 & 0 \\ 0 & \operatorname{sech} \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2$$

(3) Compound angle formula for $\sinh k$

$$\sinh k(\alpha + \beta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \cosh k \alpha \sinh k \beta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) + \cosh k \beta \sinh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)$$

$$\begin{pmatrix} 0 & \sinh(\alpha + \beta) & 0 \\ \sinh(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \sinh \beta & 0 \\ \sinh \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \sinh \alpha & 0 \\ \sinh \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(4) Compound angle formula for $\cosh k$

$$\cosh k(\alpha + \beta) = \cosh k \alpha \cosh k \beta + \sinh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \sinh k \beta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)$$

$$\begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \sinh \alpha & 0 \\ \sinh \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sinh \beta & 0 \\ \sinh \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(5) Compound angle formula for $\tanh k$

$$\tanh k(\alpha + \beta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \frac{\tanh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) + \tanh k \beta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)}{1 + \tanh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \tanh k \beta (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)}$$

$$\begin{pmatrix} 0 & \tanh(\alpha + \beta) & 0 \\ \tanh(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tanh \alpha + \tanh \beta & 0 \\ \tanh \alpha + \tanh \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 + \tanh \alpha \tanh \beta & 0 & 0 \\ 0 & 1 + \tanh \alpha \tanh \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

(6) Double angle formula for $\sinh k$

$$\sinh k(2\alpha)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = 2 \cosh k \alpha \sinh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)$$

$$\begin{pmatrix} 0 & \sinh 2\alpha & 0 \\ \sinh 2\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \sinh \alpha & 0 \\ \sinh \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(7) Double angle formula for $\cosh k$

$$\cosh k(2\alpha) = \cosh^2 k \alpha + \left(\sinh k \alpha (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \right)^2$$

$$\begin{pmatrix} \cosh 2\alpha & 0 & 0 \\ 0 & \cosh 2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 + \begin{pmatrix} 0 & \sinh \alpha & 0 \\ \sinh \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2$$

(8) Double angle formula for **tanhk**

$$\mathbf{tanhk}(2\alpha)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \left(2\mathbf{tanhk}\alpha(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \right) \left(1 + (\mathbf{tanhk}\alpha(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times))^2 \right)^{-1}$$

$$\begin{pmatrix} 0 & \tanh 2\alpha & 0 \\ \tanh 2\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & \tanh \alpha & 0 \\ \tanh \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 + \tanh^2 \alpha & 0 & 0 \\ 0 & 1 + \tanh^2 \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

(9) Products of **sinhk**

$$2\mathbf{sinhk}\alpha(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \mathbf{sinhk}\beta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \mathbf{coshk}(\alpha + \beta) - \mathbf{coshk}(\alpha - \beta)$$

$$2 \begin{pmatrix} 0 & \sinh \alpha & 0 \\ \sinh \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sinh \beta & 0 \\ \sinh \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \cosh(\alpha - \beta) & 0 & 0 \\ 0 & \cosh(\alpha - \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(10) Products of **coshk**

$$2\mathbf{coshk}\alpha \mathbf{coshk}\beta = \mathbf{coshk}(\alpha + \beta) + \mathbf{coshk}(\alpha - \beta)$$

$$2 \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \cosh(\alpha - \beta) & 0 & 0 \\ 0 & \cosh(\alpha - \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(11) Products of **sinhk** and **coshk**

$$2\mathbf{sinhk}\alpha(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \mathbf{coshk}\beta = \mathbf{sinhk}(\alpha + \beta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) + \mathbf{sinhk}(\alpha - \beta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)$$

$$2 \begin{pmatrix} 0 & \sinh \alpha & 0 \\ \sinh \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \sinh(\alpha + \beta) & 0 \\ \sinh(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sinh(\alpha - \beta) & 0 \\ \sinh(\alpha - \beta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The axis osculating hyperbolic complex number for the plane orthogonal to the $\hat{j}$ -hat rotation axis is $\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times$. The relations for yz plane ($\hat{j} \times$ rotation axis) hyperbolic basis sines, cosines, and tangents (**sinhj**, **coshj**, and **tanhj**) are

(A) Hyperbolic Euler form

$$\mathbf{coshj}\theta + \mathbf{sinhj}\theta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = \mathbf{exphj}\theta$$

(B) Hyperbolic Conjugate

$$\mathbf{coshj}\theta - \mathbf{sinhj}\theta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = \mathbf{exphj}(-\theta)$$

(1) Basic hyperbolic relation

$$\mathbf{coshj}^2\theta - (\mathbf{sinhj}\theta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times))^2 = 1$$

(2) Basic hyperbolic relation

$$1 - (\mathbf{tanhj}\theta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times))^2 = \mathbf{sechj}^2\theta$$

(3) Compound angle formula

$$\mathbf{sinhj}(\alpha + \beta)(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = \mathbf{coshj}\alpha \mathbf{sinhj}\beta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) + \mathbf{coshj}\beta \mathbf{sinhj}\alpha(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)$$

(4) Compound angle formula

$$\mathbf{coshj}(\alpha + \beta) = \mathbf{coshj}\alpha \mathbf{coshj}\beta + \mathbf{sinhj}\alpha(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) \mathbf{sinhj}\beta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)$$

(5) Compound angle formula

$$\mathbf{tanhj}(\alpha + \beta)(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = \frac{\mathbf{tanhj}\alpha(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) + \mathbf{tanhj}\beta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)}{1 + \mathbf{tanhj}\alpha(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) \mathbf{tanhj}\beta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)}$$

(6) Double angle formula

$$\mathbf{sinhj}(2\alpha)(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = 2\mathbf{coshj}\alpha \mathbf{sinhj}\alpha(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)$$

(7) Double angle formula

$$\mathbf{coshj}(2\alpha) = \mathbf{coshj}^2\alpha + (\mathbf{sinhj}\alpha(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times))^2$$

(8) Double angle formula

$$\mathbf{tanhj}(2\alpha)(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = \left(2\mathbf{tanhj}\alpha(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) \right) \left(1 + (\mathbf{tanhj}\alpha(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times))^2 \right)^{-1}$$

(9) Products of **sinhj**

$$2\mathbf{sinhj}\alpha(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) \mathbf{sinhj}\beta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = \mathbf{coshj}(\alpha + \beta) - \mathbf{coshj}(\alpha - \beta)$$

(10) Products of **coshj**

$$2\coshj\alpha\coshj\beta = \coshj(\alpha + \beta) + \coshj(\alpha - \beta)$$

(11) Products **sinhj** and **coshj**

$$2\sinhj\alpha(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)\coshj\beta = \sinhj(\alpha + \beta)(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) + \sinhj(\alpha - \beta)(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)$$

Diadically expanded into matrix base for the j -hat principal rotation axis:nb, **sinhj** is coupled with $\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times$

$$\sinhj\theta = \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sinh\theta \end{pmatrix} \quad \sinhj\theta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sinh\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sinh\theta \\ 0 & 0 & 0 \\ \sinh\theta & 0 & 0 \end{pmatrix}$$

(A) Hyperbolic Euler form for **cishj** = **exphj**

$$\coshj\theta + \sinhj\theta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = \text{exphj } \theta$$

$$\begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix} + \begin{pmatrix} 0 & 0 & \sinh\theta \\ 0 & 0 & 0 \\ \sinh\theta & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}$$

(B) Hyperbolic Conjugate Euler form

$$\coshj\theta - \sinhj\theta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = \text{exphj } (-\theta)$$

$$\begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix} - \begin{pmatrix} 0 & 0 & \sinh\theta \\ 0 & 0 & 0 \\ \sinh\theta & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh\theta & 0 & -\sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}$$

(1) Basic hyperbolic relation for **sinhj** and **coshj**

$$\coshj^2\theta - (\sinhj\theta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times))^2 = 1$$

$$\begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix}^2 - \begin{pmatrix} 0 & 0 & \sinh\theta \\ 0 & 0 & 0 \\ \sinh\theta & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) Basic hyperbolic relation for **tanhj** and **sechj**

$$1 - (\tanhj\theta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times))^2 = \text{sechj}^2\theta$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \tanh\theta \\ 0 & 0 & 0 \\ \tanh\theta & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} \text{sech}\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \text{sech}\theta \end{pmatrix}^2$$

(3) Compound angle formula for **sinhj**

$$\sinhj(\alpha + \beta)(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = \coshj\alpha\sinhj\beta(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) + \coshj\beta\sinhj\alpha(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)$$

$$\begin{pmatrix} 0 & 0 & \sinh(\alpha + \beta) \\ 0 & 0 & 0 \\ \sinh(\alpha + \beta) & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & \sinh\beta \\ 0 & 0 & 0 \\ \sinh\beta & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cosh\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\beta \end{pmatrix} \begin{pmatrix} 0 & 0 & \sinh\alpha \\ 0 & 0 & 0 \\ \sinh\alpha & 0 & 0 \end{pmatrix}$$

(4) Compound angle formula for **coshj**

$$\mathbf{coshj}(\alpha + \beta) = \mathbf{coshj}\alpha \mathbf{coshj}\beta + \mathbf{sinhj}\alpha (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) \mathbf{sinhj}\beta (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)$$

$$\begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} \cosh\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\alpha \end{pmatrix} \begin{pmatrix} \cosh\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\beta \end{pmatrix} + \begin{pmatrix} 0 & 0 & \sinh\alpha \\ 0 & 0 & 0 \\ \sinh\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \sinh\beta \\ 0 & 0 & 0 \\ \sinh\beta & 0 & 0 \end{pmatrix}$$

(5) Compound angle formula for **tanhj**

$$\mathbf{tanhj}(\alpha + \beta)(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = \frac{\mathbf{tanhj}\alpha (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) + \mathbf{tanhj}\beta (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)}{1 + \mathbf{tanhj}\alpha (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) \mathbf{tanhj}\beta (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)}$$

$$\begin{pmatrix} 0 & 0 & \tanh(\alpha + \beta) \\ 0 & 0 & 0 \\ \tanh(\alpha + \beta) & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \tanh\alpha + \tanh\beta \\ 0 & 0 & 0 \\ \tanh\alpha + \tanh\beta & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 + \tanh\alpha \tanh\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + \tanh\alpha \tanh\beta \end{pmatrix}^{-1}$$

(6) Double angle formula for **sinhj**

$$\mathbf{sinhj}(2\alpha)(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = 2\mathbf{coshj}\alpha \mathbf{sinhj}\alpha (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)$$

$$\begin{pmatrix} 0 & 0 & \sinh 2\alpha \\ 0 & 0 & 0 \\ \sinh 2\alpha & 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} \cosh\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & \sinh\alpha \\ 0 & 0 & 0 \\ \sinh\alpha & 0 & 0 \end{pmatrix}$$

(7) Double angle formula for **coshj**

$$\mathbf{coshj}(2\alpha) = \mathbf{coshj}^2\alpha + (\mathbf{sinhj}\alpha (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times))^2$$

$$\begin{pmatrix} \cosh 2\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh 2\alpha \end{pmatrix} = \begin{pmatrix} \cosh\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\alpha \end{pmatrix}^2 + \begin{pmatrix} 0 & 0 & \sinh\alpha \\ 0 & 0 & 0 \\ \sinh\alpha & 0 & 0 \end{pmatrix}^2$$

(8) Double angle formula for **tanhj**

$$\mathbf{tanhj}(2\alpha)(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = (2\mathbf{tanhj}\alpha (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)) (1 + (\mathbf{tanhj}\alpha (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times))^2)^{-1}$$

$$\begin{pmatrix} 0 & 0 & \tanh 2\alpha \\ 0 & 0 & 0 \\ \tanh 2\alpha & 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & 0 & \tanh\alpha \\ 0 & 0 & 0 \\ \tanh\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 + \tanh^2\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + \tanh^2\alpha \end{pmatrix}^{-1}$$

(9) Products of **sinhj**

$$2\mathbf{sinhj}\alpha (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) \mathbf{sinhj}\beta (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = \mathbf{coshj}(\alpha + \beta) - \mathbf{coshj}(\alpha - \beta)$$

$$2 \begin{pmatrix} 0 & 0 & \sinh\alpha \\ 0 & 0 & 0 \\ \sinh\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \sinh\beta \\ 0 & 0 & 0 \\ \sinh\beta & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} - \begin{pmatrix} \cosh(\alpha - \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh(\alpha - \beta) \end{pmatrix}$$

(10) Products of **coshj**

$$2\cosh j\alpha \cosh j\beta = \cosh j(\alpha + \beta) + \cosh j(\alpha - \beta)$$

$$2 \begin{pmatrix} \cosh \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \alpha \end{pmatrix} \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \beta \end{pmatrix} = \begin{pmatrix} \cosh(\alpha + \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} + \begin{pmatrix} \cosh(\alpha - \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh(\alpha - \beta) \end{pmatrix}$$

(11) Products of **sinhj** and **coshj**

$$2\sinh j\alpha (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) \cosh j\beta = \sinh j(\alpha + \beta) (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) + \sinh j(\alpha - \beta) (\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)$$

$$2 \begin{pmatrix} 0 & 0 & \sinh \alpha \\ 0 & 0 & 0 \\ \sinh \alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh \beta \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sinh(\alpha + \beta) \\ 0 & 0 & 0 \\ \sinh(\alpha + \beta) & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \sinh(\alpha - \beta) \\ 0 & 0 & 0 \\ \sinh(\alpha - \beta) & 0 & 0 \end{pmatrix}$$

The axis osculating hyperbolic complex number for the plane orthogonal to the $\hat{i}$ -hat rotation axis is $\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times \hat{k}$. The relations for zx plane ($\hat{i} \times$ rotation axis) hyperbolic basis sines, cosines, and tangents (**sinhi**, **coshi**, and **tanhi**) are

- (A) Hyperbolic Euler form $\cosh i\theta + \sinh i\theta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = \exp i\theta$
- (B) Hyperbolic Conjugate $\cosh i\theta - \sinh i\theta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = \exp i(-\theta)$
- (1) Basic hyperbolic relation $\cosh i^2\theta - (\sinh i\theta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times))^2 = 1$
- (2) Basic hyperbolic relation $1 - (\tanh i\theta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times))^2 = \operatorname{sech} i^2\theta$
- (3) Compound angle formula $\sinh i(\alpha + \beta) (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = \cosh i\alpha \sinh i\beta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) + \cosh i\beta \sinh i\alpha (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)$
- (4) Compound angle formula $\cosh i(\alpha + \beta) = \cosh i\alpha \cosh i\beta + \sinh i\alpha (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) \sinh i\beta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)$
- (5) Compound angle formula $\tanh i(\alpha + \beta) (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = \frac{\tanh i\alpha (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) + \tanh i\beta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)}{1 + \tanh i\alpha (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) \tanh i\beta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)}$
- (6) Double angle formula $\sinh i(2\alpha) (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = 2\cosh i\alpha \sinh i\alpha (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)$
- (7) Double angle formula $\cosh i(2\alpha) = \cosh i^2\alpha + (\sinh i\alpha (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times))^2$
- (8) Double angle formula $\tanh i(2\alpha) (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = (2\tanh i\alpha (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)) (1 + (\tanh i\alpha (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times))^2)^{-1}$
- (9) Products of **sinhi** $2\sinh i\alpha (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) \sinh i\beta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = \cosh i(\alpha + \beta) - \cosh i(\alpha - \beta)$
- (10) Products of **coshi** $2\cosh i\alpha \cosh i\beta = \cosh i(\alpha + \beta) + \cosh i(\alpha - \beta)$
- (11) Products **sinhi** and **coshi** $2\sinh i\alpha (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) \cosh i\beta = \sinh i(\alpha + \beta) (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) + \sinh i(\alpha - \beta) (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)$

Diadically expanded into matrix base for the $\hat{i}$ -hat principal rotation axis:

nb, **sinhi** is coupled with $\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times$

$$\sinh i\theta = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sinh \theta & 0 \\ 0 & 0 & \sinh \theta \end{pmatrix} \quad \sinh i\theta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sinh \theta & 0 \\ 0 & 0 & \sinh \theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh \theta \\ 0 & \sinh \theta & 0 \end{pmatrix}$$

(A) Hyperbolic Euler form for **cishi** = **exphi**

$$\mathbf{coshi} \theta + \mathbf{sinhi} \theta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = \mathbf{exphi} \theta$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & \cosh \theta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh \theta \\ 0 & \sinh \theta & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & \sinh \theta \\ 0 & \sinh \theta & \cosh \theta \end{pmatrix}$$

(B) Hyperbolic Conjugate Euler form

$$\mathbf{coshi} \theta - \mathbf{sinhi} \theta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = \mathbf{exphi} (-\theta)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & \cosh \theta \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh \theta \\ 0 & \sinh \theta & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & -\sinh \theta \\ 0 & -\sinh \theta & \cosh \theta \end{pmatrix}$$

(1) Basic hyperbolic relation for **sinhi** and **coshi**

$$\mathbf{coshi}^2 \theta - (\mathbf{sinhi} \theta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times))^2 = 1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & \cosh \theta \end{pmatrix}^2 - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh \theta \\ 0 & \sinh \theta & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) Basic hyperbolic relation for **tanhi** and **sechi**

$$1 - (\mathbf{tanhi} \theta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times))^2 = \mathbf{sechi}^2 \theta$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tanh \theta \\ 0 & \tanh \theta & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \operatorname{sech} \theta & 0 \\ 0 & 0 & \operatorname{sech} \theta \end{pmatrix}^2$$

(3) Compound angle formula for **sinhi**

$$\mathbf{sinhi}(\alpha + \beta) (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = \mathbf{coshi} \alpha \mathbf{sinhi} \beta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) + \mathbf{coshi} \beta \mathbf{sinhi} \alpha (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh(\alpha + \beta) \\ 0 & \sinh(\alpha + \beta) & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & \cosh \alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh \beta \\ 0 & \sinh \beta & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & \cosh \beta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh \alpha \\ 0 & \sinh \alpha & 0 \end{pmatrix}$$

(4) Compound angle formula for **coshi**

$$\mathbf{coshi}(\alpha + \beta) = \mathbf{coshi} \alpha \mathbf{coshi} \beta + \mathbf{sinhi} \alpha (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) \mathbf{sinhi} \beta (\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & \cosh \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & \cosh \beta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh \alpha \\ 0 & \sinh \alpha & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh \beta \\ 0 & \sinh \beta & 0 \end{pmatrix}$$

(5) Compound angle formula for **tanhi**

$$\mathbf{tanhi}(\alpha + \beta)(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = \frac{\mathbf{tanhi} \alpha(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) + \mathbf{tanhi} \beta(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)}{1 + \mathbf{tanhi} \alpha(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)\mathbf{tanhi} \beta(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tanh(\alpha + \beta) \\ 0 & \tanh(\alpha + \beta) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tanh \alpha + \tanh \beta \\ 0 & \tanh \alpha + \tanh \beta & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + \tanh \alpha \tanh \beta & 0 \\ 0 & 0 & 1 + \tanh \alpha \tanh \beta \end{pmatrix}^{-1}$$

(6) Double angle formula for **sinhi**

$$\mathbf{sinhi}(2\alpha)(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = 2\mathbf{coshi} \alpha \mathbf{sinhi} \alpha(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh 2\alpha \\ 0 & \sinh 2\alpha & 0 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & \cosh \alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh \alpha \\ 0 & \sinh \alpha & 0 \end{pmatrix}$$

(7) Double angle formula for **coshi**

$$\mathbf{coshi}(2\alpha) = \mathbf{coshi}^2 \alpha + (\mathbf{sinhi} \alpha(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times))^2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh 2\alpha & 0 \\ 0 & 0 & \cosh 2\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & \cosh \alpha \end{pmatrix}^2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh \alpha \\ 0 & \sinh \alpha & 0 \end{pmatrix}^2$$

(8) Double angle formula for **tanhi**

$$\mathbf{tanhi}(2\alpha)(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = (2\mathbf{tanhi} \alpha(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)) (1 + (\mathbf{tanhi} \alpha(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times))^2)^{-1}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tanh 2\alpha \\ 0 & \tanh 2\alpha & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tanh \alpha \\ 0 & \tanh \alpha & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + \tanh^2 \alpha & 0 \\ 0 & 0 & 1 + \tanh^2 \alpha \end{pmatrix}^{-1}$$

(9) Products of **sinhi**

$$2\mathbf{sinhi} \alpha(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) \mathbf{sinhi} \beta(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = \mathbf{coshi}(\alpha + \beta) - \mathbf{coshi}(\alpha - \beta)$$

$$2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh \alpha \\ 0 & \sinh \alpha & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh \beta \\ 0 & \sinh \beta & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\alpha - \beta) & 0 \\ 0 & 0 & \cosh(\alpha - \beta) \end{pmatrix}$$

(10) Products of **coshi**

$$2\mathbf{coshi} \alpha \mathbf{coshi} \beta = \mathbf{coshi}(\alpha + \beta) + \mathbf{coshi}(\alpha - \beta)$$

$$2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \alpha & 0 \\ 0 & 0 & \cosh \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & \cosh \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\alpha + \beta) & 0 \\ 0 & 0 & \cosh(\alpha + \beta) \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\alpha - \beta) & 0 \\ 0 & 0 & \cosh(\alpha - \beta) \end{pmatrix}$$

(11) Products of **sinhi** and **coshi**

$$2\mathbf{sinhi}\alpha(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)\mathbf{coshi}\beta = \mathbf{sinhi}(\alpha + \beta)(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) + \mathbf{sinhi}(\alpha - \beta)(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)$$

$$2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh \alpha \\ 0 & \sinh \alpha & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \beta & 0 \\ 0 & 0 & \cosh \beta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh(\alpha + \beta) \\ 0 & \sinh(\alpha + \beta) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sinh(\alpha - \beta) \\ 0 & \sinh(\alpha - \beta) & 0 \end{pmatrix}$$

Because the hyperbolic complex number $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$ is so large and bulky, and so unwieldy [it doubles the equation size in some instances], I had considered the commutator notation $[\hat{i} \times, \hat{j} \times]$. The problem here is that the commutator notation changes the sign of the second term whereas the hyperbolic complex number requires the same sign on both terms for the hyperbolic properties. With the sign change as the commutator operator is currently defined, this produces that for the circular complex number $\hat{k} \times$

$$[\hat{i} \times, \hat{j} \times] = \hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times = \hat{k} \times$$

So this notation isn't as of this time used here. It may, though, be of value to other readers in their research and mathematical toolkit. The three circular complex numbers (basis rotation operators $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$) in commutator operator notation are

$$[\hat{i} \times, \hat{j} \times] = \hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times = \hat{k} \times$$

$$[\hat{k} \times, \hat{i} \times] = \hat{k} \times \hat{i} \times - \hat{i} \times \hat{k} \times = \hat{j} \times$$

$$[\hat{j} \times, \hat{k} \times] = \hat{j} \times \hat{k} \times - \hat{k} \times \hat{j} \times = \hat{i} \times$$

In diadically expanded matrix bases

$$[\hat{i} \times, \hat{j} \times] = \hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{k} \times$$

$$[\hat{k} \times, \hat{i} \times] = \hat{k} \times \hat{i} \times - \hat{i} \times \hat{k} \times = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \hat{j} \times$$

$$[\hat{j} \times, \hat{k} \times] = \hat{j} \times \hat{k} \times - \hat{k} \times \hat{j} \times = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \hat{i} \times$$

This demonstrates the overwhelming importance and non-triviality of a single sign as it completely changes the nature of the operators (here, from circular to hyperbolic). It definitely would have made nice notation for the axis osculating hyperbolic basis functions. Thinking of maybe a possible conjugate notation $[\hat{i} \times, \hat{j} \times]^*$, $[\hat{k} \times, \hat{i} \times]^*$, and $[\hat{j} \times, \hat{k} \times]^*$

$$[\hat{i} \times, \hat{j} \times]^* = \hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[\hat{k} \times, \hat{i} \times]^* = \hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$[\hat{j} \times, \hat{k} \times]^* = \hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Also, defining the circular and hyperbolic complex numbers (basis function cross products with the same properties of complex algebra [i.e., vector algebra]) in this way, such as $\hat{i} \times \hat{j} \times \pm \hat{j} \times \hat{i} \times$ with **(1)** $\hat{k} = \hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times$ for circular and **(2)** $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$ for hyperbolic, provides a means to produce elliptic harmonic (with $a\hat{i} \times \hat{j} \times - b\hat{j} \times \hat{i} \times$) and elliptic hyperbolic (with $a\hat{i} \times \hat{j} \times + b\hat{j} \times \hat{i} \times$) complex numbers which I have yet to type up. Works out quite nicely as well. I have too many results and not enough time to type them all up... and I haven't started with the calculus yet either [which I am expecting will correspond with electromagnetism one for one].